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SMALL DENOMINATORS. I. CONCERNING THE REPRESENTATION OF A CIRCLE

by

V.N. Arnol'd

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SMALL DENOMINATORS. I.

Concerning the Representation of a Circle

by V. N. Arnol'd

It was shown in the first part of this book that the analytical transformation of a circle, which differs little from a turn, whose number of rotations is irrational and which satisfies certain arithmetical requirements, can be converted to a turn by an analytical change of variable. Discussed in the second part is the space of circle representations and the place occupied in that space by various types of representations. References are made in the appendices to the investigation of trajectories on a torus and to Dirichlet's problem for the equation of a vibrating string.

Introduction

The continuous representations of a circle were studied by Poincare (see 8, chapter XV, pp. 165-191) in connection with a qualitative investigation of trajectories on a torus. Dirichlet's problem for the equation of a vibrating string is also conducive to such representations, but a topological investigation in this case is found to be insufficient (see [5]). Outlined in the first part of this work is an attempt to provide an analytical definition of Denjoy's theorem [2] which completes Poincare's theory.

Let us assume that F(z) represents a real periodic function $F(z + 2\pi) = F(z)$ on a real axis and an analytical function, and that $F'(z) \neq -1$ with Im z = 0. Then the representation of the complex plane region $z \to Az = z + F(z)$ will correspond to the orientation-retaining geomorphism B of the circle points $w(z) = e^{iz}$:

$$w = w(z) \to w(Az) \equiv Bw.$$

In this sense we can state that A is the analytical representation of a circle.

Let us assume that the number of rotations* A equals $2\pi\mu$. It follows from Denjoy's theorem that when μ is irrational, there exists a

*It is assumed that the reader is familiar with the results of the work [1] (pp. 165-191 and 322-335) and [2] which are included in the textbooks [3] (pp. 65-76) and [4] (pp. 442-456).

continuous reversible real function $\phi(z)$ of a real z, and it is periodic in a sense that

$$\varphi(z+2\pi)=\varphi(z)+2\pi,$$

and that

$$\varphi(Az)=\varphi(z)+2\pi\mu. \qquad (1)$$

We shall say that ϕ is a new parameter, and that in the q parameter the transformation of A becomes a turn to angle $2\pi\mu$. There can be only one such function ϕ (correct to an additive constant). /22

It was shown in §1 that in the case of certain irrational μ, regardless even of the analyticity of F(z), the function ϕ in (1) may not be found to be absolutely continuous. The idea of this example consists in the following. Since the rotations of a circle do not affect the length, the reduction of a transformation to a turn by an appropriate selection of a parameter amounts to finding an invariant measure of transformation. In the case of a rational rotation number, the invariant measure is concentrated, as a rule, in separate points, the points of the transformation cycles. However, if the rotation number is irrational but approximate to the rational, the invariant measure retains its singular nature even though it is closely distributed all around the circle.

The following hypothesis appears plausible:

There exists such a set as $M \subseteq [0,1]$ of measure 1, whereby the solutions of equation (1) for each $\mu \in M$, under any analytical transformation of A with a rotation number $2\pi\mu$, are analytical.

So far this has been proved only in regard to analytical transformations (§4, theorem 2)* which are fairly close to a turn to angle $2\pi\mu$. The proof is in the method of solving equation (1) by way of the following equation:

$$g(z+2\pi\mu)-g(z)=f(z). \qquad (2)$$

*Notation in proofreading. The work of A. Finzi [38], [39] came to the attention of the author during the printing of this article. It follows from work [38] that if the rotation number of a fairly smooth representation of a circle satisfies certain arithmetical requirements, the rotation can be converted to a turn by a continuously differentiable change of variables. Thus the A. Finzi method does not require the transformation to be close to a turn; this is partially confirmed by the above hypothesis. A. Finzi points out, however, that he sees no possibility of using his methods in cases requiring a very smooth change of variables. This article contains a partial answer to some of his questions; the reader will find a partial answer to some of the questions raised here in the mentioned articles by A. Finzi.

The solution of this equation with the aid of Fourier's series reveals a number of small denominators complicating convergence. The calculation of the successive corrections designed to adapt the solution of equation (2) to equation (1) is made by a Newton-type method, and the rapid convergence of this method makes it possible to realize not only all the approximations of the perturbation theory but also the limit transition.

The Newton method was used for such a purpose by A.N. Kolmogorov [6]. Theorem 2 of this article is a kind of discrete analog of his theorem of the preservation of conditional periodic motion with a little change in the Hamilton function. Unlike the work [6], we have no analytical integral invariant but are looking for it. Moreover, we are proving (in theorem 2) the analyticity of the dependence on the small parameter ϵ which implies the convergence of series by power ϵ which is usual in the theory of perturbation.

Direct proof of the convergence of these series cannot be provided, and in this connection A. N. Kolmogorov even advanced the hypothesis of their divergence (prior to his study of K. L. Siegel's work [7]).* /23

Another hypothesis expressed in A. N. Kolmogorov's report [8] proved to be true; the problems involving small denominators are associated with the monogenic Borel functions [9]. With reference to our case, this was established in §7, 8 and is used in §11.

Some important problems involving small denominators were solved by C. L. Siegel (see [7, 33, 34, 35]). The Schroeder equation has direct reference to the representation of a circle: is it possible to use the analytical change of variables $\phi(z) = z + b_2z^2 + \ldots$ in order to convert the representation of a zero neighborhood in a complex plane, determined by the analytical function $f(z) = c^{2\pi i\mu} z + a_2z^2 + \ldots$ to a turn to angle $2\pi\mu$.

The result achieved by Siegel [7] is similar to our theorem 2, and can be obtained by the same method. The problem of a center is a special aspect of the representation of a circle whose radius in some instances is equal to zero. Here the situation is simpler, as compared to the general aspect, since the solution (Schroeder's series) can be formally expressed at once. The use of the Newton method also produces Schroeder's series; unlike theorem 2, each coefficient of the solution will be accurately determined after the finite number of approximations.

The second part of the article contains a classification of the representations of a circle and a discussion of the typical nature of various cases. In §9, function $\mu(T)$ (the rotation number) is introduced

*In his report to the Moscow mathematical society, 13 Jan. 1959.

in the space of the circle representation. This is followed by a study of the rational (§10) and irrational (§11) level μ from the point of view of their arrangement (theorems 6 and 7) and massiveness (theorems 5 and 8). Rough representations are topologically overwhelming from the viewpoint of A. A. Andronov and L. S. Pontryagin [10]. With normal cycles and a rational rotation number, they form an open absolutely dense set.* Typical also from the point of view of dimension in the finite-dimensional subspace is the ergodic case. A two-dimensional subspace of representations $x \to x + a + \epsilon \cos x$ is discussed in §12.

The preceding results are applied in §§13 and 14 to a qualitative investigation of trajectories on the torus and to Dirichlet's problem for the equation of a vibrating string.

The author expresses his gratitude to A. N. Kolmogorov for his valuable advice and assistance.

PART I

Concerning the Analytical Representation of a Circle

The gist of the first part of the article is contained in §§4-6 (theorem 2). To understand the proof of theorem 2 (§§5 & 6), we need the subparagraphs 2.1, 2.3 of §2 and 3.3 of §3. The implicit function and finite-increase lemmas may be referred to as needed. Each of the §§1, 2, 7 can be read independently of all the rest. The generalization of theorem 2 (theorem 3), used in the second part of this work, is proved in §8. /24

§1. When Is a New Parameter Not an Absolutely Continuous Function of an Old One?

1.1. This paragraph deals with the analytical transformation A of circle C, circle subset G_n (n = 1,2, ...) and natural number N_n (n = 1,2, ...) in such a way that:

1. mes $G_n \to 0$ with $n \to \infty$.
2. $A^{N_n} (C \backslash G_n) \subset G_n$.
3. The rotation number of transformation A is irrational.

*Notation in proofreading. This result was obtained also by V. A. Pliss in an article [43] published during the printing of this work.

This transformation A cannot be converted to a turn by an absolutely continuous change of variable. Indeed, let us assume that ϕ is a continuous parameter in which transformation A is changed to a turn to angle $2\pi\mu$ (Denjoy's theorem assumes the existence of ϕ). The powers of A are also converted to turns. Let us assume that $G \subset C$. The measure of set $\phi(G)$ of values $\phi(x)$, $x \in G$ is congruent to measure $\phi(A^N G)$, so that these sets combine during the turn. It therefore follows from condition 2 that:

$$2\pi - \text{mes}\,\varphi(G_n) \leqslant \text{mes}\,\varphi(G_n)$$

and

$$\text{mes}\,\varphi(G_n) \geqslant \pi.$$

In view of condition 1, ϕ is not an absolutely continuous function on C.

1.2. The following lemmas are used in the construction.

LEMMA α. Let us assume that A is a forward* semistable analytical representation of a circle in the neighborhood of the real axis, and let points z_o, $z_k = A(z_{k-1})(0 < k < n)$ form a cycle--that is, $A(z_{n-1})$. Then for any $\epsilon > 0$ in the mentioned neighborhood of the real axis there is a transformation A which differs from A' by less than ϵ and which has exactly one cycle, namely $z_o, z_1, \ldots, z_{n-1}$.

Proof. Let us make an analytical correction $\Delta(z)$ in the area under consideration which turns to zero at the points $z_o, \ldots, z_{n-1}$, and a positive correction at the remaining real points.

We shall assume

$$A'(z) = A(z) + \varepsilon'\Delta(z);$$

when the $\epsilon' > 0 |\epsilon'\Delta(z)| < \epsilon$ value in the mentioned area is low, $A'(z)$ also represents the transformation of a circle. Obviously, the transformation $(A')^n$ will shift all points z forward not less than transformation A^n, thereby displacing the points $z_o, \ldots, z_{n-1}$ by $2\pi m$ and the remaining points by more than $2\pi m$; lemma α has been proved.

Definition. Let A be a transformation of circle C, and G a set on it. We shall say that transformation A possesses property 2 in relation to G and N, if $A^N(C \setminus G) \subset G$. /25

LEMMA β. The transformation A with a single cycle $z_o, \ldots, z_{n-1}$, with any $\epsilon > 0$, possesses property 2 in relation to the set G_ϵ of points in the ϵ-neighborhood of the cycle and any N exceeding a specified $N_o(\epsilon)$.

*This means that with certain integers m, n and any real z, $A^n(z) \geq z + 2\pi\mu$, and an equality is achieved.

Proof. Let $z_i > x < z_j$, where $z_i\, z_j$ is one of the arcs into which the cycle divides the circle. Points $A^{kn}(x)(k = 1, 2, \ldots)$ lie on arc $z_i\, z_j$ and form a monotonic sequence (for more details see §10). Hence, if transformation A has a forward semistability (the case of a backward semistability is fully analogical),

$$A^{kn}(x) \xrightarrow[k \to +\infty]{} z_j.$$

Indeed, let λ be the limit of the monotonic sequence $A^{kn}(x)$; then λ is an invariant in relation to A^n and belongs to the cycle that satisfies the inequalities

$$z_i < \lambda \leqslant z_j.$$

Thus,

$$\lim_{k \to \infty} A^{kn+l}(x) = A^l(z_j).$$

The same is true of the other intervals into which the circle is divided by the cycle.

Let us examine the points $x_i = z_i + \epsilon$. According to the proof, all the points $A^N x_i$, beginning with some $N_0(\epsilon)$, lie in the ϵ-neighborhood of the cycle. Obviously, this N_0 is the unknown quantity.

LEMMA γ. Let transformation A possess property 2 in relation to G and N, and let $\epsilon > 0$. Then there exists such $\delta > 0$ that any transformation B, differing from A by less than δ, possesses property 2 in relation to N and the ϵ-neighborhood of G.

Proof. The lemma obviously follows from the continuous dependence of A^N on A.

LEMMA δ. Let A be a forward semistable transformation, $B(z) = A(z) + h$, $h > 0$. Then the rotation number μ of transformation B is definitely larger than the rotation number $\frac{m}{n}$ of transformation A.

Proof. Obviously, $\mu \geq \frac{m}{n}$. Here $B^n(z) > A^n(z)$ and therefore B has no cycle of the n order. Hence, $\mu > \frac{m}{n}$.

LEMMA ϵ (a singular case of Liuville's theorem). If the inequality $|\alpha - \frac{m}{n}| < \frac{c}{|n|}$, with any $c > 0$, has an infinite set of irreducible solutions $\frac{m}{n}$, the number α is irrational.

Proof. If $\alpha = \frac{p}{q}$, then with $n > q$

$$\left|\frac{p}{q}-\frac{m}{n}\right|>\frac{\frac{1}{|q|}}{|n|},$$

as the fraction m/n is irreducible, and that means that $|pn - qm| \neq 0$ with $q < n$.

1.3 Transformation A is constructed as the limit of the transformation sequence A_n with rational rotation numbers. We shall begin with the transformation $z \to A_1(z)$; we shall assume that it possesses the following properties: /26

1_1. A_1 is analytical in the band Im $z < R$ and in this band $A_1(z) < \frac{C}{2}$.

2_1. The rotation number of A_1 is rational: $\mu_1 = \frac{p_1}{q_1}$.

3_{1a}. A_1 is semistable (forward).

3_{1b}. A_1 has exactly one cycle.

The existence of such A_1 is obvious: by a proper $h > 0$ selection, $A_1' = A_1 + h$ with properties 1_1, 2_1 and 3_1 can be obtained from any A_1'', and A_1' can then be corrected to A_1 according to lemma α. The following transformations A_n can be achieved from the preceding ones by the process based on the following.

INDUCTIVE LEMMA. Let $\delta_n > 0$ and let us assume that the given transformations $A_k (k = 1,2,\ldots,n)$ and $R > 0$, $C > 0$ are such that

1_n. With $|\text{Im } z| < R$, A_k are analytical and satisfy the inequalities

$$|A_k(z) - A_{k-1}(z)| < \frac{C}{2^k} \quad (A_0(z) \equiv 0).$$

2_n. The rotation number of A_k is rational also with $k > 1$

$$\left|\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}}\right| < \frac{1}{(k-1)^2 (\max_{l<k} q_l)^2}.$$

3_n. A_k are forward semistable, each having a single cycle.

It is then possible to construct the transformation A_{n+1} in such a way that the sequence $A_k (k = 1,2,\ldots,(n + 1)$ will possess the properties 1_{n+1}, 2_{n+1}, 3_{n+1} and 4_{n+1}.

$$|A_{n+1}(z) - A_n(z)| < \delta_n \text{ npu } \operatorname{Im} z = 0.$$

Proof. Let us examine the transformations $A_\lambda: z \to A_n(z) + \lambda$, $\lambda > 0$. Obviously there exists $\lambda_0 > 0$ such that when $\lambda < \lambda_0$

$$|A_\lambda(z) - A_n(z)| < \frac{C}{2^{n+2}} \quad (|\operatorname{Im} z| < R),$$

$$|A_\lambda(z) - A_n(z)| < \frac{\delta_n}{2} \quad (\operatorname{Im} z = 0)$$

and the rotation number A_λ is definitely larger than $\frac{p_n}{q_n}$ (lemma δ) and smaller than

$$\frac{p_n}{q_n} + \frac{1}{n^2 (\max\limits_{i<n} q_j)^2}$$

(for continuity of rotation number, see §9). Let rotation number A_{λ_0} be μ; we shall select the rational number $\frac{p_{n+1}}{q_{n+1}}$,

$$\frac{p_n}{q_n} < \frac{p_{n+1}}{q_{n+1}} < \mu,$$

and from among all the λ, whose rotation number A_λ is $\frac{p_{n+1}}{q_{n+1}}$, we shall select the largest, say, λ_1. The transformation A_{λ_1} possesses the properties 1_{n+1}, 2_{n+1}, 4_{n+1} and, as can readily be seen, is forward semi-stable. We shall apply lemma α to it; then we shall get transformation A_{n+1} which satisfies all the requirements of the inductive lemma. /27

1.4. Transformation A satisfies the conditions 1_1, 2_1, 3_1 of the inductive lemma with the same C, R. We shall describe the selection δ_n by transferring the induction from A_n to A_{n+1}. We shall designate the ϵ-neighborhood of the single cycle A_n as G_n^*, where $\epsilon > 0$ is such that the measure G_n^* is less than 2^{-n-2}. According to lemma β, a N_n will be found, whereby A_n possesses property 2 in relation to G_n and N_n. According to lemma γ, there exists $\delta_n^* > 0$, whereby transformation A possesses property 2 in relation to N_n and G_n, of the G_n-neighborhood, measure 2^{-n-1}, if on the real axis. We shall select

$$|A(z) - A_n(z)| < \delta_n^*.$$

$$\delta_{n+1} = \min\left(\frac{\delta_n}{2}, \frac{\delta_n^*}{2}\right)$$

(we formally consider $\delta_0 = 0$). Applying the inductive lemma, we get A_{n+1}.

If the transformations $A_n (n = 1,2,\ldots)$ are constructed by the described method, then, in view of property 1_n, this sequence will converge uniformly in the $|\mathrm{Im}\, z| < R$ area so that limit A is an analytical transformation. Obviously,

$$|A(z) - A_n(z)| \leqslant \sum_{k=n}^{\infty} |A_{k+1}(z) - A_k(z)| \leqslant \sum_{k=n}^{\infty} \delta_n \left(\frac{1}{2}\right)^{n+1} \leqslant \delta_n \quad (\mathrm{Im}\, z = 0)$$

with any n, and therefore A possesses property 2 in relation to G_n and $N_n (n = 1,2,\ldots)$. We conclude from property 2_n and the continuity of the rotation number, on the basis of lemma ϵ, that rotation number A is irrational. Indeed, with any n

$$\left|\mu - \frac{p_n}{q_n}\right| \leqslant \sum_{k=n}^{\infty} \frac{1}{k^2 (\max\limits_{l<k} q_l)^2} \leqslant \sum_{k=n}^{\infty} \frac{1}{k^2 q_n^2} < \frac{2}{q_n^2}.$$

Thus, all three properties indicated in 1.1 are fulfilled, so that A represents the sought for transformation.

1.5. Remark. Examining the structure of the example, it is easy to see that the transformation A with the mentioned properties can be found in any family of analytical transformations

$$z \to A_\Delta z \equiv z + \Delta + F(z)$$

in any neighborhood of any transformation with an irrational rotation number, if the family only possesses the following property: there are no turns among the transformations A_Δ^n. The family $z \to z + \Delta + \frac{1}{2}\cos z$ /28
probably possesses this property; in that case the example can be presented by a simple analytical formula.

§2. On the Functional Equation* $g(z + 2\pi\mu) - g(z) = f(z)$

2.1 Let f(z) be a function of period 2π, and μ a real number.

*Gilbert [12] refers to this equation as an analytical problem for which there is a nonanalytical solution. It occurs in the researches into the metric theory of dynamic systems (see [13, 11]) and represents a simple problem with small denominators.

Notation in proofreading. The mentioned article had already been submitted to the printer when the author learned of A. Wintner's famous article [40] in which the equation under discussion is studied from a contemporary point of view, apparently for the first time.

Define from equation

$$g(z+2\pi\mu)-g(z)=f(z) \qquad (1)$$

the function g(z) which has a 2π period.

Obviously, if equation (1) is insoluble

$$\int_0^{2\pi} f(z)\,dz=0.$$

Further, if g(z) is a solution, then g(z) + C is also a solution. We shall therefore consider only the right-hand sides which are on the average equal to zero, and look only for solutions which are on the average equal to zero. In each function $\phi(z)$ we shall single out a constant for $(0, 2\pi)$

$$\bar{\varphi}=\frac{1}{2\pi}\int_0^{2\pi}\varphi(z)\,dz$$

and a variable

$$\tilde{\varphi}(z)=\varphi(z)-\bar{\varphi}.$$

Thus the condition required for the solution of equation (1) is the equality $\bar{f}=0$; hereafter, the solution (1) will always imply the variable part g(z).

If $\mu=\frac{m}{n}$, i.e., it is rational, the existence of a solution requires that

$$\sum_{k=1}^{n} f\left(z+2\pi\frac{k}{n}\right)=0,$$

as this sum is expressed by a solution in the form of

$$\sum_{k=1}^{n} g\left(z+2\pi\frac{m}{n}+2\pi\frac{k}{n}\right)-\sum_{k=1}^{n} g\left(z+2\pi\frac{k}{n}\right),$$

and the items in these two sums are the same. If such a condition is fulfilled, a solution exists but it is defined only correct to an arbitrary function of the $\frac{2\pi}{n}$ period, as such a function satisfies the homogeneous equation /29

$$g\left(z+2\pi\frac{m}{n}\right)-g(z)=0.$$

But if μ is irrational, there is only one solution, namely:

1) With μ being irrational, equation (1) cannot have two different continuous solutions.

Proof. The difference between two continuous solutions to equation (1) satisfies the following equations:

$$g(z+2\pi)-g(z)=0,$$
$$g(z+2\pi\mu)-g(z)=0,$$

that is, this continuous function has two incommensurable periods. Such a function is a constant (see [15], pp. 55-56); it takes on the same value at all $2\pi k + 2\pi\mu l$ points which form a solid set everywhere. As

$$\int_0^{2\pi} g(z)\,dz=0,$$

the mentioned constant is zero.

2) With μ being irrational, equation (1) cannot have two measurable solutions which are almost always incongruent.

Proof. Let us take another look at the difference between the two solutions of function g(z). It may be considered as a function on the circle, as it has a period 2π. According to the condition,

$$g(z+2\pi\mu)-g(z)=0,$$

that is, g(z) does not change during the turn to angle $2\pi\mu$. Therefore the set E_a of the points on the circle, where $g(z) > a$, is invariant in relation to the turn to angle $2\pi\mu$. If the function g(z) is (almost everywhere) constant, such a constant (as in the case of 1), is zero. If g(z) is not a constant, the set E_a will have a measure of $0 < \text{mes } E_a < 2\pi$ at a specified value a. But it is well known that a set which is invariant in relation to a turn to an angle incommensurable with 2π has a measure of zero or a full measure (see [3], for example; proof can be adduced by the mere use of the theorem of the density point). Thus $g(z) = 0$ (almost everywhere).

The expansion of function f(z) in a Fourier series

$$f(z)=\sum_{n\neq 0} f_n e^{inz},$$

produces the following Fourier coefficients g(z):

$$g_n e^{2\pi i\mu n} - g_n = f_n,$$

that is,

$$g_n = \frac{f_n}{e^{2\pi i\mu n} - 1}, \quad g(z) = \sum_{n \neq 0} g_n e^{inz}. \tag{2}$$

With μ being rational, some of the denominators become zero. When μ is irrational, there are numerous small denominators among the denominators. We shall point out that /30

$$|e^{2\pi\mu n} - 1| > |\mu n - m| \tag{3}$$

with any integer n or a specified integer m. The smallness of the denominators in (2) therefore depends on the approximations of μ by rational numbers.

LEMMA 1 (see [16]). Let $\epsilon > 0$. For almost every μ, $0 \leq \mu \leq 1$ (in point of the Lebesgue measure) there is $K > 0$, such that

$$|\mu n - m| \geqslant \frac{K}{n^{1+\epsilon}} \tag{4}$$

with any integers m and $n > 0$.

Proof. We shall select some $K > 0$ and estimate the measure of set E_k of points μ, $0 < \mu < 1$, which do not satisfy the inequality (4) and which we shall rewrite as

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^{2+\epsilon}}.$$

This set contains all the points $\frac{m}{n}$ with the neighborhood of radius $\frac{K}{n^{2+\epsilon}}$. At a fixed value n, the number of these points will be n + 1, and the total length of the neighborhoods (of 0, 1) will be $\frac{K}{n^{1+\epsilon}}$. Therefore

$$\operatorname{mes} E_K \leqslant \sum_{n=1}^{\infty} \frac{K}{n^{1+\epsilon}} = c(\epsilon) K.$$

The set of points μ, for which the number K required in the lemma does not exist, is included in E_k with any $K > 0$, and its measure therefore is less than $c(\epsilon)K$ with any K--that is, it equals zero.

2.2 We will show that in almost all μ the small denominators have an insignificant effect on the convergence of series (2).

LEMMA 2 (see [17]). The series

$$S=\sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}}\frac{1}{|n\mu-m_n|} \tag{5}$$

is convergent at any $\epsilon < 0$ and any integers m_n, if μ is such that

$$|\mu n-m|\geqslant\frac{K}{n^{1+\epsilon-\delta}}\quad (K>0,\quad 0<\delta<\epsilon) \tag{6}$$

with all integers m and $n > 0$.

Proof. It may be assumed, without disrupting the overall relationship, that $|\mu n - m_n| < 1$. Let us examine the series S_i, of the same type as S, in which the summation extends only to the indices $n = n_k^{(i)}$ for which

$$\frac{1}{2^{i+1}}\leqslant|\mu n_k^{(i)}-m_{n_k^{(i)}}|<\frac{1}{2^i}\quad (i=0,1,2,\ldots;\ n_{k+1}^{(i)}>n_k^{(i)}). \tag{7}$$

In the aggregate, the series S_i contain all the S members, so that all that needs to be shown is that

$$\sum_{i=0}^{\infty}S_i<\infty.$$

To estimate S_i, we shall point out that, in view of (6), the consecutive numbers $n_k^{(i)}$, $n_{k+1}^{(i)}$ of the series S_i terms are far removed: as the following inequality follows from (7) /31

$$|\mu(n_k^{(i)}-n_{k+1}^{(i)})-m|<\frac{1}{2^{i-1}},$$

we deduce from (6):

$$\frac{1}{2^{i-1}}>\frac{K}{N_i^{1+\epsilon-\delta}},$$

where

$$N_i=\min_{0<k<\infty}(n_{k+1}^{(i)}-n_k^{(i)}).$$

Hence we get:

$$N_i > (2^{i-1}K)^{\frac{1}{1+\varepsilon-\delta}}. \tag{8}$$

It is obvious that $n_1^{(i)} > N_i$, and that in general $n_k^{(i)} > kN_i$, so that in view of (5), (7) and (8) we have:

$$S_i < \sum_{k=1}^{\infty} \frac{2^{i+1}}{(kN_i)^{1+\varepsilon}} = \frac{2^{i+1}}{N_i^{1+\varepsilon}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\varepsilon}} = \frac{2^{i+1}}{2^{(i-1)\frac{1+\varepsilon}{1+\varepsilon-\delta}}} L(\varepsilon, K) \quad (L(\varepsilon, K) > 0).$$

$$S_i < 2^{1+\frac{1+\varepsilon}{1+\varepsilon-\delta}} L 2^{i\left(1-\frac{1+\varepsilon}{1+\varepsilon-\delta}\right)} = L'(\varepsilon, \delta, K)\theta^i.$$

Here

$$\theta = 2^{1-\frac{1+\varepsilon}{1+\varepsilon-\delta}} < 1,$$

therefore

$$\sum_{i=0}^{\infty} S_i < \infty,$$

which completes the proof.

It is a known fact that if $f(x)$ is a function differentiable $p+\epsilon$ times,* its Fourier coefficients are of a decreasing order

$$f_n = O\left(\frac{1}{n}\right)^{p+\varepsilon},$$

and if

$$f_n = O\left(\frac{1}{n}\right)^{p+1+\varepsilon},$$

the $f(x)$ is differentiable $p+\epsilon$ times. In view of that, we get the following result from inequality (3) and lemmas 1 and 2 applied to series (2):

If the function $f(z)$ is differentiable $p + 1 + \epsilon + \delta$ times, then

*That is a function in which the p^{th} derivative fulfills the Goelder condition of the ϵ power: $|f^{(p)}(x + h) - f^{(p)}(x)| < Ch^{\epsilon}$.

equation (1) has a solution which is differentiable p + ϵ times in almost every case of μ.

On the other hand, it is not difficult to illustrate that when the number μ can be well approximated by rational numbers, series (2) converges slowly, or is not convergent at all, despite the rapidly decreasing numerators f_n. Therefore, even if f(z) is analytical, we may find cases when g(z) is not analytical but infinitely differentiable, or differentiable only a finite number of times, or only continuous or even discontinuous, or that the solution is immeasurable (see [14, 17]).* /32

2.3. Let us examine equation (1) in a class of analytical functions. Investigating this case, we shall recall the two lemmas dealing with the Fourier coefficients of analytical functions.

LEMMA 3. If the function f(z) of period 2μ in the area $|\mathrm{Im}\, z| < R$ is analytical, and in this area $|f(z)| \leq C$, then its Fourier coefficients satisfy the inequalities

$$|f_n| \leqslant Ce^{-|n|R}.$$

Proof. According to the definition,

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(z)\, e^{-inz}\, dz.$$

In view of the periodicity $f(z)e^{-inz}$,

$$\int_0^{i\tau} f(z)\, e^{-inz}\, dz = \int_{2\pi}^{2\pi+i\tau} f(z)\, e^{-inz}\, dz,$$

therefore

$$f_n = \frac{1}{2\pi}\int_{0+i\tau}^{2\pi+i\tau} f(z)\, e^{-inz}\, dz$$

with any $\tau \in |-R, R|$. Integrating in the case of $n > 0$ by a straight $\tau = -R$ and in $n < 0$ by $\tau = R$, we get:

*A. N. Kolmogorov advanced the hypothesis that the latter case can always be realized if the series $\sum_{n \neq 0} \frac{|f_n^2|}{|e^{2\pi i \mu n} - 1|^2}$ is divergent.

$$|f_n| \leqslant \frac{1}{2\pi} \int_0^{2\pi} Ce^{-|n|R} dz,$$

which completes the proof.

LEMMA 4. Let the Fourier coefficients of function f(z) fulfill the inequalities $|f_n| \leq Ce^{-|n|R}$. Then f(z) is analytical and fulfills the following inequality with $|\mathrm{Im}\, z| < R - \delta$, $0 < \delta < R$

$$|f(z)| \leqslant \frac{2C}{1-e^{-\delta}},$$

and its derivative fulfills the inequality

$$|f'(z)| \leqslant \frac{2C}{(1-e^{-\delta})^2}.$$

Proof. With $|\mathrm{Im}\, z| \leq R - \delta$, $0 < \delta < R$, it is obvious that /33

$$|e^{inz}| \leqslant e^{|n|(R-\delta)}.$$

Therefore

$$|f_n e^{inz}| \leqslant Ce^{-|n|\delta}$$

and

$$\sum_{n=-\infty}^{\infty} |f_n e^{inz}| \leqslant 2 \sum_{n=0}^{\infty} Ce^{-n\delta} \leqslant \frac{2C}{1-e^{-\delta}}.$$

The same applies to

$$\sum_{n=-\infty}^{\infty} |f_n ine^{inz}| \leqslant 2C \sum_{n=0}^{\infty} ne^{-n\delta} \leqslant \frac{2C}{(1-e^{-\delta})^2}.$$

In the $|\mathrm{Im}\, z| < R - \delta$ area the convergence of the series is absolutely uniform. The lemma has been proved.

It is now easy to examine the analytical solution of equation (1).

THEOREM 1. Let $f(z) = \tilde{f}(z)$ be an analytical function of period 2π and $|\mathrm{Im}\, z| < R\, |f(z)| \leq C$. Let μ be an irrational number, $K > 0$ and

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^2} \qquad (9)$$

with any integers m and $n < 0$. Then the equation

$$g(z+2\pi\mu)-g(z)=f(z)$$

has an analytical solution $g(z)=\tilde{g}(z)$, and with $|\mathrm{Im}\, z| \leq R - 2\delta$ and any $\delta < 1$, $0 < \delta < \frac{R}{2}$.

$$|g(z)| \leqslant \frac{4C}{K\delta^3}, \tag{10}$$

$$|g'(z)| \leqslant \frac{8C}{K\delta^4}. \tag{11}$$

Proof. Using function f(z) and lemma 3 to estimate the Fourier coefficients f_n, and making use of inequalities (3) and (9), we get from (2):

$$|g_n| \leqslant \frac{C}{K} n^2 e^{-|n|R}. \tag{12}$$

We note the simple inequality

$$|n|^p \leqslant \left(\frac{p}{e}\right)^p \frac{e^{|n|\delta}}{\delta^p}, \tag{13}$$

which is true with any $\delta > 0$. (Indeed, $p \ln x < p \ln\frac{p}{e} + x$, as the function x p ln x - x has a maximum at $\frac{p}{x} = 1$; assuming that $x = \delta|n|$, we get (13).) Applying (13) to (12) (with $p = 2$), we get:

$$|g_n| \leqslant \frac{Ce^{-|n|R}e^{|n|\delta}}{K\delta^2} = \frac{Ce^{-|n|(R-\delta)}}{K\delta^2},$$

hence, on the basis of lemma 4, we find in the area $|\mathrm{Im}\, z| \leq R - 2\delta$: /34

$$|g(z)| \leqslant \frac{2C}{K\delta^2(1-e^{-\delta})}, \quad |g'(z)| \leqslant \frac{2C}{K\delta^2(1-e^{-\delta})^2}.$$

Since $\delta < 1$ $|1 - e^{-\delta}| > \frac{\delta}{2}$, we find from this the inequalities (10), (11). The theorem has been proved.

Remark 1. Obviously, if f(z) on a real straight line is real, then the solution is also real.

Remark 2. If the function $f(z, \lambda)$ is analytically dependent on parameter λ, the solution (in term of theorem 1) is also analytical with regard to the parameter.

2.4. Let us examine equation (1) with complex μ. In this case the solution of the homogeneous equation

$$g(z+2\pi\mu)-g(z)=0$$

can be any double periodic function with periods 2π and $2\pi\mu$, and this is therefore not the sole solution of the problem. If g(z) is required to be analytical in an area wider than $|\mathrm{Im}\ 2\pi\mu|$, solution (1) can be determined by a simple number correct to a constant. Actually, such a wide area contains a parallelogram of periods, and the solution of a homogeneous equation in it is limited on the entire plane--that is, it is a constant. The condition $\bar{g}=0$ yields a single solution which is produced by series (2). This series is convergent under any complex μ, but we are interested in estimates, and the neighborhoods of rational μ should therefore be excluded. We shall designate as M_k^r the set of points in a rectangle on a complex plane $0\leq \mathrm{Re}\mu\leq 1$, $|\mathrm{Im}\,\mu|\leq r$ so that with all integers m, n the following inequality is fulfilled

$$\left|\mu-\frac{m}{n}\right|\geqslant\frac{K}{|n|^3}.$$

Obviously, M_k^r includes $\bar{\mu}$, $1-\mu$, $1-\bar{\mu}$ along with μ.

Instead of inequality (3) we have:

$$|e^{2\pi iz}-1|\geqslant\min\left(\frac{1}{2},\pi|z-m|\right) \tag{14}$$

for any complex z with some integer m. We shall prove inequality (14). If $|e^{2\pi iz}-1|\geq\frac{1}{2}$, then (14) is proved. If $|e^{2\pi iz}-1|<\frac{1}{2}$, we shall connect points 1 and $e^{2\pi iz}$ with the segment and examine the integral

$$\frac{1}{2\pi i}\int_1^{e^{2\pi iz}}\frac{dw}{w}=\frac{1}{2\pi i}(\ln e^{2\pi iz}-\ln 1)=z-m,$$

where lnw is one of the logarithm branches and $\ln 1=2\pi im$ (m is an integer). As the integration segment lies entirely within the circle

$$|w-1|<\frac{1}{2},$$

and in this circle $|w|>\frac{1}{2}$, then /35

$$\left|\int_1^{e^{2\pi iz}}\frac{dw}{w}\right|\leqslant 2|e^{2\pi iz}-1|.$$

Therefore

$$|z - m| \leqslant \frac{1}{\pi} |e^{2\pi i z} - 1|,$$

which completes the proof.

If $\mu \in M_k^r$, then by applying (14) to $z = \mu n$, we find that

$$|e^{2\pi i \mu n} - 1| \geqslant \min\left(\frac{1}{2}, \frac{\pi K}{n^2}\right).$$

Thus, if $\mu \in M_k^r$, where $K < \frac{1}{2\pi}$, then

$$|e^{2\pi i \mu n} - 1| \geqslant \frac{\pi K}{n^2}. \tag{15}$$

THEOREM 1'. Let $f(z) = \tilde{f}(z)$ be an analytical function of period 2π and $|\text{Im } z| \leq R|f(z)| \leq C$, and let $\mu \in M_k^r$, $K < \frac{1}{2\pi}$. Then the equation

$$g(z + 2\pi\mu) - g(z) = f(z) \tag{1}$$

will have an analytical solution $g(z) = \tilde{g}(z)$, and with $|\text{Im } (z - 2\pi\mu)| < R - 2\delta$ and any $\delta < 1$, $0 < \delta < \frac{R}{2}$,

$$|g(z)| \leqslant \frac{4C}{\pi K \delta^3}, \quad |g'(z)| \leqslant \frac{8C}{\pi K \delta^4}. \tag{16}$$

Proof. According to formula (2) and lemma 3, we have:

$$|g_n e^{inz}| \leqslant \frac{Ce^{-|n|R}}{e^{2\pi i \mu n} - 1} e^{in(z - 2\pi\mu + 2\pi\mu)}, \tag{17}$$

and with $|\text{Im}(z - 2\pi\mu)| < R - 2\delta$

$$|e^{in(z-2\pi\mu)}| < e^{|n|(R-2\delta)},$$

so that it follows from (17):

$$|g_n e^{inz}| \leqslant \frac{Ce^{-2\delta|n|}}{1 - e^{-2\pi i \mu n}}.$$

Since $1 - \mu \in M_k^r$, in accordance with (15),

$$|1 - e^{-2\pi i \mu n}| \geqslant \frac{\pi K}{n^2}.$$

which means that

$$|g_n e^{inz}| \leqslant \frac{Ce^{-2\delta|n|}n^2}{\pi K}.$$

This, in view of (13), results in the convergence of the series g(z) and g'(z) and, consequently, also the truth of inequalities (16) (see proof of theorem 1 and lemma 4).

Remark 1. Note 2 to theorem 1 is applicable also to theorem 1'.

Remark 2. We shall fix function f and number z and examine the dependence of the found solution on μ: /36

$$g(\mu) = \sum_{n \neq 0} \frac{f_n}{e^{2\pi i\mu n} - 1} e^{inz}. \tag{2}$$

The function $g(\mu)$ is analytical in the upper and lower semiplane but the axis Im $\mu = 0$ is an excision. Series (2) converges on it almost everywhere, but to a discontinuous limit. This will not prevent us from differentiating the solution by μ in §7, even with Im $\mu = 0$, by making use of Borel's ideas [9]. In the meantime, we believe that the formula

$$\frac{\partial g}{\partial \mu} = -\sum_{n \neq 0} \frac{2\pi i n e^{2\pi i\mu n} f_n}{(e^{2\pi i\mu n} - 1)^2} e^{inz}$$

makes sense only in the upper and lower semiplanes separately.

§3. The Lemmas Required To Prove Theorem 2

3.1. LEMMA 5. If function f(z) is analytical in each point of segment $z_1 z_2$ and $\left|\frac{dt}{dz}\right| \leq L$, then $|f(z_2) - f(z_1)| \leq L|z_2 - z_1|$.

Proof. Indeed,

$$f(z_2) - f(z_1) = \int_{z_1}^{z_2} \frac{df(z)}{dz} dz,$$

hence

$$|f(z_2) - f(z_1)| \leqslant \int_{z_1}^{z_2} \left|\frac{df(z)}{dz}\right| |dz| \leqslant L|z_2 - z_1|.$$

Remark. The example $f(z) = e^{iz}$, $z_1 = 0$, $z_2 = 2\pi$ shows that in a complex region the theorem of the finite increment in the form of

$$f(z_2) - f(z_1) = \frac{df(\xi)}{dz}(z_2 - z_1)$$

or

$$|f(z_2) - f(z_1)| = \left|\frac{df(\xi)}{dz}\right| |z_2 - z_1|$$

is incorrect.

3.2. LEMMA 6 (concerning an implicit function). Let functions $F(\epsilon)$, $\Phi(\epsilon, \Delta)$ be analytical, and with $|\epsilon| \leq \epsilon_0$, $|\Delta| \leq \Delta_0$

$$|F(\varepsilon)| \leqslant M_1, \quad |\Phi(\varepsilon, \Delta)| \leqslant M_2 |\Delta|,$$

where $\frac{M_1}{1 - M_2} < \frac{\Delta_0}{3}$ and $M_2 < \frac{1}{6}$. Then

1. The equation $\Delta + F(\epsilon) + \Phi(\epsilon, \Delta) = 0$ has an analytical solution of $\Delta^*(\epsilon)$ which, with $|\epsilon| < \epsilon_0$, fulfills the inequality $|\Delta^*(\epsilon)| \leq \frac{M_1}{1 - M_2}$.

2. The equation $\Delta + F(\epsilon) + \Phi(\epsilon, \Delta) = \Delta_1$ has a root $\Delta = \Delta(\Delta, \epsilon)$ analytically dependent on Δ_1 and ϵ, $|\Delta_1| < \frac{\Delta_0}{6}$, $|\epsilon| < \epsilon_0$, root $\Delta = \Delta(\Delta_1, \epsilon)$, and

$$|\Delta(\Delta_1, \varepsilon) - \Delta^*(\varepsilon)| \leqslant 2|\Delta_1|.$$

Proof. 1. The circle $|\Delta| < \frac{M_1}{1 - M_2}$ is at $\frac{M_1}{1 - M_2} < \Delta_0$. $|\epsilon| < \epsilon_0$ in the region where $|F(\epsilon)| \leq M_1$, $|\Phi(\epsilon, \Delta)| < M_2|\Delta|$, and the transformation $\Delta \to -F(\epsilon) - \Phi(\epsilon, \Delta)$ therefore puts it within itself: /37

$$|F(\varepsilon) + \Phi(\varepsilon, \Delta)| \leqslant M_1 + \frac{M_1}{1 - M_2} M_2 = \frac{M_1}{1 - M_2}.$$

The fixed transformation point represents the sought for solution of $\Delta^*(\epsilon)$; analyticity follows from the ordinary theorem of the implicit function, as

$$\frac{\partial}{\partial \Delta}(\Delta + F(\varepsilon) + \Phi(\varepsilon, \Delta)) \neq 0,$$

which follows from the estimation of $\frac{\partial \phi}{\partial \Delta}$ with the aid of the Cauchy integral: with $|\Delta| \leq \frac{2\Delta_0}{3}$, $|\epsilon| < \epsilon_0$

$$\left|\frac{\partial\Phi}{\partial\Delta}\right| \leqslant \frac{M_2\Delta_0}{\frac{\Delta_0}{3}} < \frac{1}{2}.$$

2. In the representation of $w \rightarrow w + \Phi(w,\epsilon)$, point $\Delta^*(\epsilon)$ changes to $F(\epsilon)$, and points w of circle $|w - \Delta^*(\epsilon)| \leq 2|\Delta_1|$ to points

$$w + \Phi(\Delta^*(\varepsilon), \varepsilon) + [\Phi(w, \varepsilon) - \Phi(\Delta^*(\varepsilon), \varepsilon)].$$

Since under the terms of the lemma applying to the points of this circle

$$|\Phi(w, \varepsilon) - \Phi(\Delta^*(\varepsilon), \varepsilon)| \leqslant |\Delta_1|$$

(lemma 5), the image of circle $|w - \Delta^*(\epsilon)| \leq 2|\Delta_1|$ contains the entire circle $|w + F(\epsilon)| \leq \Delta_1$ and has point $\Delta(\Delta_1,\epsilon)$ which changes to $\Delta_1 - F(\epsilon)$. This point fulfills inequality

$$|\Delta - \Delta^*| \leqslant 2|\Delta_1|$$

and equation

$$\Delta = \Delta_1 - F(\varepsilon) - \Phi(\varepsilon, \Delta).$$

Unity and analyticity follow from inequality $\left|\frac{\partial\Phi}{\partial\Delta}\right| < \frac{1}{2}$.

Remark. It is easy to see that if under the terms of lemma 6 the functions $F(\epsilon)$ and $\Phi(\epsilon,\Delta)$ are real, given real ϵ, Δ, then $\Delta^*(\epsilon)$ and $\Delta(\Delta_1,\epsilon)$ are real with real Δ_1, ϵ.

3.3 The Newton method (see [18, 6]). Let us assume that a solution is sought for equation $f(x) = 0$ (Fig. 1). We shall define x roughly as x_0 and find intersection point x_1 of tangential at x_0 to curve $y = f(x)$ with axis x:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

We shall further define in succession

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

and estimate the velocity of the convergence process.* Let x be the

*No exact premises or estimates are given here. They are cited in work [18] in very general terms which do not, however, cover the discussions appearing in the following paragraphs.

unknown solution and $|x_0 - x| = \epsilon$. Then the deviation of the curve from the line tangential to it, at point x_0, will have an order of ϵ^2 at point x, which means that $|x_1 - x|$ is a magnitude of the order ϵ^2. Thus, after the n-th step the error will be ϵ^{2n}--a very rapid convergence. /38

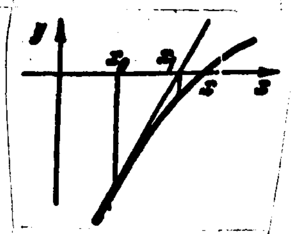

Fig. 1

We shall apply a method of the Newton method type to the solution of the linear functional equation approximable by the equation discussed in §2. The rapid convergence will paralyze the small denominators appearing on every step.

§4. Theorem 2 and Basic Lemma

4.1. Leading considerations. The transformation

$$z \to z + 2\pi\mu$$

is a turn of the circle. The transformation

$$z \to z + 2\pi\mu + \varepsilon F(z)$$

is a turn, disturbed by a member $\epsilon F(z)$, which is small together with ϵ. Its rotation number, even if $\bar{F} = 0$, may be different from $2\pi\mu$. However, it is possible to find $\Delta = \Delta(\epsilon)$ such that the transformation

$$z \to z + 2\pi\mu + \Delta + \varepsilon F(z)$$

will have a rotation number equal to $2\pi\mu$. We shall show that in the case of μ, formally approximated by rational numbers, and fairly small ϵ

1) $\Delta(\epsilon)$ is analytically dependent on ϵ;

2) The transformation $z \to z + 2\pi\mu + \Delta + \epsilon F(z)$ can be converted to a turn to angle $2\pi\mu$ by an analytical change of variable $\phi(z) = z + g(z)$.

Here g(z) is a small correction together with ϵ, and property 2) means that

$$\varphi(z + 2\pi\mu + \Delta(\varepsilon) + \varepsilon F(z), \varepsilon) = \varphi(z, \varepsilon) + 2\pi\mu,$$

or, which is the same (the g dependence on ϵ is implied),

$$g(z + 2\pi\mu + \Delta + \varepsilon F(z)) - g(z) = -\Delta - \varepsilon F(z). \qquad (1)$$

This equation differs from the one discussed in §2 only by small magnitudes of the second order, and it is therefore natural in the first approximation to select $\Delta = \Delta_1(\epsilon)$ such that the right side of equation (1) is on the average equal zero:

$$\Delta_1 = -\varepsilon\bar{F}$$

and look for $g_1(z)$ as a solution to the equation

$$g_1(z + 2\pi\mu) - g_1(z) = -\varepsilon\bar{F}(z).$$

The g_1 defined here has an order of ϵ, and in the variable $\phi_1 = z + g_1$ our transformation

$$z \to z + 2\pi\mu + \Delta_1(\varepsilon) + \varepsilon F(z)$$

looks like this:

$$\begin{gathered}\varphi_1(z + 2\pi\mu + \Delta_1(\varepsilon) + \varepsilon F(z)) = z + 2\pi\mu + \Delta_1 + \varepsilon F + \\ + g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) = z + g_1(z) + 2\pi\mu + \\ + [g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) - g_1(z + 2\pi\mu)] + \\ + [g_1(z + 2\pi\mu) - g_1(z) + \varepsilon\tilde{F}(z)] + (\Delta_1 + \varepsilon\bar{F}).\end{gathered}$$

Thanks to the selection of Δ_1 and $g_1(z)$, the last two terms become zero, and we get: /39

$$\varphi_1(z) \to \varphi_1(z) + 2\pi\mu + F_2(z, \varepsilon).$$

The "perturbance" will now look like this:

$$F_2(z, \varepsilon) = g_1(z + 2\pi\mu + \Delta_1 + \varepsilon F) - g_1(z + 2\pi\mu) = \frac{dg_1(\xi)}{dz}(\Delta_1 + \varepsilon F).$$

Here $\frac{dg_1}{dz}$, like g_1, is a magnitude of the order ϵ and, as it is also related to the second cofactor, the perturbance in parameter ϕ_1 has an order of ϵ^2. The transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + F_2$$

can be treated the same way to determine the "frequency correction" Δ_2 and the new parameter ϕ_2 so that the transformation

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2$$

in the parameter ϕ_2 becomes the following transformation

$$\varphi_2 \to \varphi_2 + 2\pi\mu + F_3,$$

where $F_3 \sim \epsilon^4$. In this case, however, the transformation in parameter z

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_3$$

will not look like the following:

$$z \to z + 2\pi\mu + \Delta + \varepsilon F.$$

We must therefore begin with the transformation

$$z \to z + 2\pi\mu + \Delta_1(\varepsilon) + \Delta_1'(\Delta_2) + \varepsilon F;$$

then with the appropriate selection of Δ_1' (Δ_2) it will be possible to get the following transformation in parameter ϕ_1

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_2 + F_2'(\varphi_1),$$

and the following in parameter ϕ_2

$$\varphi_2 \to \varphi_2 + 2\pi\mu + F_3',$$

and so on. The rapid convergence of the $(F_n \sim \epsilon^{2^{n-1}})$ method makes it possible to realize a limit transition, and find a new parameter $\phi(z, \epsilon)$ and a final correction $\Delta(\epsilon)$ possessing the properties 1) and 2) within that limit. The usual method employed in the theory of perturbation for the solution of our problem would be to look for $\Delta(\epsilon)$ and $\phi(z, \epsilon)$ in the form of series by power ϵ, and determine the coefficients of the series successively from the fulfillment of equation (1) in the first approximation, the second and so on. The convergence of such series cannot be proved by direct estimates but it is borne out by the basic theorem of this work cited below.

4.2. THEOREM 2. Let a given family of analytical transformations of a circle be analytically dependent on the two parameters ϵ, Δ

$$z \to A(z, \varepsilon, \Delta) \equiv z + 2\pi\mu + \Delta + F(z, \varepsilon) \tag{2}$$

and the numbers $R > 0$, $\epsilon_1 > 0$, $K > 0$, $L > 0$ are such that

1) $F(z + 2\pi, \epsilon) = F(z, \epsilon)$;

2) With $\mathrm{Im}\ z = \mathrm{Im}\ \epsilon = 0$ always $\mathrm{Im}\ F(z, \epsilon) = 0$;

3) With $|\mathrm{Im}\ z| \leq R$, $|\epsilon| \leq \epsilon_0$

$$|F(z, \varepsilon)| < L|\varepsilon|; \tag{3}$$

4) The irrational number μ, with any m and n integers, fulfills the inequality /40

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3}. \qquad (4)$$

Then there exist numbers ϵ' and R', $0 < \epsilon' \geq \epsilon_0$, $0 \quad R' \leq R$, and functions $\Delta(\epsilon)$, $\phi(z, \epsilon)$ which are real, given real ϵ and z, and analytical with $|\epsilon| < \epsilon'$, $|\text{Im } z| < R'$ such that

$$\varphi(A(z, \varepsilon, \Delta(\varepsilon)), \varepsilon) = \varphi(z, \varepsilon) + 2\pi\mu. \qquad (5)$$

This theorem is proved in §6 on the basis of the following lemma.

BASIC LEMMA. Let a given family of analytical transformations of a circle be analytically dependent on the two parameters ϵ, Δ

$$z \to A_0(z, \varepsilon, \Delta) \equiv z + 2\pi\mu + \Delta + F(z, \varepsilon) + \Phi(z, \varepsilon, \Delta) \qquad (6)$$

and the number $R_0 > 0$, $\epsilon_0 > 0$, $K > 0$, $\delta > 0$, $C > 0$, $0 < \Delta_0 < 1$ are such that

1) $F(z + 2\pi, \epsilon) = F(z, \epsilon)$, $\Phi(z + 2\pi, \epsilon, \Delta) = \Phi(z, \epsilon, \Delta)$;

2) With $\text{Im } z = \text{Im } \epsilon = \text{Im } \Delta = 0$ always $\text{Im } F = \text{Im } \Phi = 0$;

3) With $|\text{Im } z| \leq R_0$, $|\epsilon| \leq \epsilon_0$, $|\Delta| \leq \Delta_0$

$$|F(z, \varepsilon)| \leqslant C < \delta^8, \qquad (7)$$

$$|\Phi(z, \varepsilon, \Delta)| < \delta|\Delta|; \qquad (8)$$

4) The irrational number μ, given any integers m and n, fulfills the inequality (4);

5) The number δ fulfills the inequalities

$$\delta < \frac{K}{64}, \quad \delta < \frac{R_0}{8}, \qquad (9)$$

$$\delta < \frac{1}{36}, \qquad (10)$$

and also

$$C < \frac{\Delta_0}{6}. \qquad (11)$$

Then the existing analytical functions $z(\phi, \epsilon)$, $\Delta(\Delta_1, \epsilon)$, $F_1(\phi, \epsilon)$, $\Phi_1(\phi, \epsilon, \Delta_1)$ are such that

1. The following is identical:

$$z[A_1(\varphi, \varepsilon, \Delta_1), \varepsilon] = A_0[z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)], \tag{12}$$

where

$$A_1(\varphi, \varepsilon, \Delta_1) \equiv \varphi + 2\pi\mu + \Delta_1 + F_1(\varphi, \varepsilon) + \Phi_1(\varphi, \varepsilon, \Delta_1). \tag{13}$$

2. $F_1(\phi + 2\pi, \epsilon) = F_1(\phi, \epsilon)$, $\Phi_1(\phi + 2\pi, \epsilon, \Delta_1) = \Phi_1(\phi, \epsilon, \Delta_1)$; $z(\phi + 2\pi, \epsilon) = z(\phi, \epsilon) + 2\pi$.

3. With $\mathrm{Im}\,\phi = \mathrm{Im}\,\Delta_1 = \mathrm{Im}\,\epsilon = 0$, always $\mathrm{Im}\,z = \mathrm{Im}\,\Delta = \mathrm{Im}\,F_1 = \mathrm{Im}\,\Phi_1 = 0$.

4. With $|\Delta_1| \leq C$, $|\mathrm{Im}\,\phi| \leq R_0 - 7\delta$, $|\epsilon| \leq \epsilon_0$

$$|F_1(\varphi, \varepsilon)| \leqslant \frac{C^2}{\delta^4}, \tag{14}$$

$$|\Phi_1(\varphi, \varepsilon, \Delta_1)| \leqslant \delta^2 |\Delta_1|, \tag{15}$$

$$|z(\varphi, \varepsilon) - \varphi| \leqslant \frac{C}{\delta^4}, \quad \left|\frac{\partial z}{\partial \varphi}\right| < 2, \tag{16}$$

$$|\Delta(\Delta_1, \varepsilon)| \leqslant \Delta_0, \quad \left|\frac{\partial \Delta}{\partial \Delta_1}\right| < 2. \tag{17}$$ /41

The basic lemma shows that a small perturbance (of the order of C) of the turn $z \to z + 2\pi\mu$ can be compensated for by changing the parameter $z \to \phi$, with $\Delta = \Delta(\Delta_1, \epsilon)$ in such a way that the difference from the turn in the new parameters is of the order of C^2. The proof of the lemma is given in the following paragraph.

4.3. We are making use of the following assertion in §11.

Corollary of theorem 3. Let the irrational number μ fulfill inequality (4) of theorem 2, and let $R > 0$. There exists $C(R, K) > 0$ such that if the transformation

$$Az : z \to z + 2\pi\mu + F(z)$$

has rotation number $2\pi\mu$ and $|F(z)| \leq C$ with $|\mathrm{Im}\,z| \leq R$, then Az can be converted to a turn to angle $2\pi\mu$ by an analytical change of a variable.

Proof. Let us examine function

$$F_1(z) = \frac{F(z)}{\max\limits_{|\operatorname{Im} z|<R} |F(z)|}$$

and transformation family

$$A_\varepsilon z : z \to z + 2\pi\mu + \varepsilon F_1(z),$$

which fulfill the conditions of theorem 2, with $L = 1$, as $|F_1(z)| \leq 1$ with $|\operatorname{Im} z| \leq R$. According to theorem 2, there exists $\epsilon'(R, K) > 0$ such that with $\epsilon < \epsilon'$ the transformation

$$z \to z + 2\pi\mu + \Delta(\varepsilon) + \varepsilon F_1(z)$$

can be converted to a turn to angle $2\pi\mu$. We shall select $C(R, K) < \epsilon'$. Then if $|F(z)| \geq C$ with $|\operatorname{Im} z| \leq R$, there exists Δ such that

$$z \to z + 2\pi\mu + \Delta + F(z)$$

can be converted to a turn to angle $2\pi\mu$, by an analytical transformation of a coordinate, because

$$F(z) = \max_{|\operatorname{Im} z|<R} |F(z)| F_1(z),$$

and

$$\max |F(z)| \leqslant C < \varepsilon'.$$

But the rotation number Az is equal $2\pi\mu$, hence $\Delta = 0$ (see point 2 of the proof of theorem 4 in §10 which shows that regardless of how small Δ is, the rotation number of transformation $x \to z + 2\pi\mu + \Delta + F(z)$ is greater than $2\pi\mu$). The corollary has been proved.

The corollary can also be affirmed directly by the use of a construction similar to that of theorem 2. In view of the lack of parameters ϵ, Δ, this construction will be less ponderous.

4.4. Remark on a Multidimensional Case. All the constructions of §§2-8 may be understood as multidimensional by replacing a point of the circle with a point of the torus of k measurements. Condition 4) of theorem 2 is replaced by the following condition of "incommensurability" for vector $\vec{\mu}$: /42

$$|n_0 + (\vec{\mu}, \vec{n})| \geqslant \frac{K}{|\vec{n}|^\omega} \tag{18}$$

with any integral vector $\vec{n} = (n_0, \ldots, n_k)$. Here $(\vec{\mu}, \vec{n})$ are the scalar product

$$\sum_{i=1}^{k} \mu_i n_i, \quad |\vec{n}| = \sum_{i=0}^{k} |n_i|.$$

With a sufficiently large magnitude ω, condition (18) is fulfilled for almost all $\vec{\mu}$ vectors.

Without dwelling at length on the formulations and proofs of all the inequalities, lemmas and theorems for a multidimensional case, we shall cite only one result.

MULTIDIMENSIONAL THEOREM 2. Let $\vec{\mu} = (\mu_1, \ldots, \mu_k)$ be a vector with incommensurable components, such that with any integral vector $\vec{n}$

$$|n_0 + (\vec{\mu}, \vec{n})| > \frac{K}{|\vec{n}|^{k+1}}.$$

Then there exists such $\epsilon(R, C, k) > 0$, that for the vector field $\vec{F}(\vec{z})$ on an analytical and fairly small torus $F(z) < \epsilon$ with $|\text{Im } \vec{z}| < R$ there will be a vector $\vec{a}$ for which the representation of the torus

$$\vec{z} \to \vec{z} + \vec{a} + \vec{F}(\vec{z})$$

is changed to

$$\vec{\varphi} \to \vec{\varphi} + 2\pi\vec{\mu}$$

by an analytical change of variables.

§5. Proof of Basic Lemma

5.1. Construction $z(\phi, \epsilon)$, $\Delta(\Delta_1, \epsilon)$, $F_1(\phi, \epsilon)$ and $\Phi_1(\phi, \epsilon, \Delta_1)$. Function $z(\phi, \epsilon)$ is constructed as an inverse function to

$$\varphi(z, \varepsilon) = z + g(z, \varepsilon), \tag{1}$$

and function $\Delta(\Delta_1, \epsilon)$ as an inverse function to $\Delta_1(\Delta, \epsilon)$. We saw in point 4.1 that these functions should be selected in such a way that the expression

$$g(A_0(z, \varepsilon, \Delta), \varepsilon) - g(z, \varepsilon) + F(z, \varepsilon) + \Delta + \Phi(z, \varepsilon, \Delta)$$

be small. Without defining $\Delta(\Delta_1, \epsilon)$ (that is, not counting Δ as an independent variable), we shall define $g^*(z, \epsilon, \Delta)$ as the solution of equation

$$g^*(z+2\pi\mu, \varepsilon, \Delta) - g^*(z, \varepsilon, \Delta) = -\tilde{F}(z, \varepsilon) - \tilde{\Phi}(z, \varepsilon, \Delta). \tag{2}$$

Expressing the transformation A_0 (see §4, formula 6) by parameter

$$\varphi^*(z, \varepsilon, \Delta) = z + g^*(z, \varepsilon, \Delta),$$

we get

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = z + 2\pi\mu + \Delta + F(z, \varepsilon) + \Phi(z, \varepsilon, \Delta) + \\ + g^*(z+2\pi\mu, \varepsilon, \Delta) + g^*(A_0(z, \varepsilon, \Delta)) - g^*(z+2\pi\mu, \varepsilon, \Delta),$$

or, transforming the right side by the use of (2), /43

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = z + g^*(z, \varepsilon, \Delta) + 2\pi\mu + \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta) + \\ + g^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] - g^*(z+2\pi\mu, \varepsilon, \Delta).$$

Thus, according to (1), we get:

$$\varphi^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] = \varphi^*(z, \varepsilon, \Delta) + 2\pi\mu + \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta) + \\ + g^*[A_0(z, \varepsilon, \Delta), \varepsilon, \Delta] - g^*(z+2\pi\mu, \varepsilon, \Delta). \tag{3}$$

We shall define $\Delta_0^*(\epsilon)$ as a solution of equation

$$\Delta_0^*(\varepsilon) + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta_0^*(\varepsilon)) = 0 \tag{4}$$

and assume that

$$g^*(z, \varepsilon, \Delta_0^*(\varepsilon)) = g(z, \varepsilon). \tag{5}$$

The new parameter $\phi(z, \epsilon)$ is now determined by equalities (5) and (1). We shall present (3) in the form of

$$\varphi[A_0(z, \varepsilon, \Delta), \varepsilon] = \varphi(z, \varepsilon) + 2\pi\mu + \Delta_1(\varepsilon, \Delta) + \hat{F}_1(z, \varepsilon) + \hat{\Phi}_1(z, \varepsilon, \Delta), \tag{6}$$

where

$$\hat{F}_1(z, \varepsilon) = g(z_{I}, \varepsilon) - g(z_{II}, \varepsilon), \tag{7}$$

$$\hat{\Phi}_1(z, \varepsilon, \Delta) = g(z_{III}, \varepsilon) - g(z_{I}, \varepsilon), \tag{8}$$

$$\Delta_1(\varepsilon, \Delta) = \Delta + \bar{F}(\varepsilon) + \bar{\Phi}(\varepsilon, \Delta), \tag{9}$$

$$z_I = z + 2\pi\mu + \tilde{F}(z, \varepsilon) + \hat{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon)), \quad (10)$$

$$z_{II} = z + 2\pi\mu, \quad (11)$$

$$z_{III} = z + 2\pi\mu + \tilde{F}(z, \varepsilon) + \Delta_1(\varepsilon, \Delta) + \tilde{\Phi}(z, \varepsilon, \Delta). \quad (12)$$

We shall determine $z(\phi, \epsilon)$ from (1), $\Delta(\Delta_1, \epsilon)$ from (9) and indicate

$$F_1(\varphi, \varepsilon) = \hat{F}_1(z(\varphi, \varepsilon), \varepsilon), \quad (13)$$

$$\Phi_1(\varphi, \varepsilon, \Delta_1) = \hat{\Phi}_1(z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)), \quad (14)$$

$$A_1(\varphi, \varepsilon, \Delta_1) = \varphi[A_0(z(\varphi, \varepsilon), \varepsilon, \Delta(\Delta_1, \varepsilon)), \varepsilon]. \quad (15)$$

5.2. We shall prove that the above constructed functions are the ones we sought. The assertions 1, 2 and 3 of the basic lemma are clearly fulfilled. The proof of assertion 4 is based on the following estimates.

1°. Estimate $\Delta_0^*(\epsilon)$. Lemma 6 (§3) can be applied to equation (4) on the basis of inequalities (10), (11) of §4. Here $M_1 = C$, $M_2 = \delta$ and as

$$\frac{C}{1-\delta} < \frac{\Delta_0}{3}, \quad \delta < \frac{1}{2}$$

(see formulas 10, 11, §4), then

$$|\Delta_0^*(\varepsilon)| < \frac{C}{1-\delta}.$$

Bearing in mind that $\delta < 1/2$, we find that with $|\epsilon| < \epsilon_0$:

$$|\Delta_0^*(\varepsilon)| < 2C. \quad (16)$$

2°. Estimate $g(z,\epsilon)$. Inequality (16) enables us to estimate the right side of equation (2). With $|\mathrm{Im}\, z| < R$, $|\epsilon| \le \epsilon_0$, $\Delta = \Delta_0^*(\epsilon)$, it follows from (16) and inequalities (7), (8), (10) of §4 that: /44

$$|\tilde{F}(z, \varepsilon) + \tilde{\Phi}(z, \varepsilon, \Delta)| \leqslant 2C + 2\delta \cdot 2C < 4C. \quad (17)$$

Applying theorem 1, §2 to equation (2), we find, on the basis of (5), (17) and condition 4) of the basic lemma, that with $|\mathrm{Im}\, z| \le R_0 - 2\delta$, $|\epsilon| \le \epsilon_0$ and any $\delta < 1$, $0 < \delta < \frac{R_0}{2}$,

$$|g(z, \varepsilon)| < \frac{8 \cdot 4C}{K\delta^3}, \quad \left|\frac{\partial g}{\partial z}\right| < \frac{16 \cdot 4C}{K\delta^4},$$

hence, in view of inequality (9), §4,

$$|g(z,\varepsilon)|<\frac{C}{\delta^4},\quad \left|\frac{\partial g(z,\varepsilon)}{\partial z}\right|<\frac{C}{\delta^5}. \tag{18}$$

As, according to inequality (7) §4, $C<\delta^8$, it follows from here that

$$|g(z,\varepsilon)|<\delta.$$

Therefore, in the representation $z\to\phi(z,\epsilon)=z+g(z,\epsilon)$ the band

$$|\mathrm{Im}\, z|\leqslant R_0-2\delta$$

will transfer to the domain containing the following band:

$$|\mathrm{Im}\,\varphi|\leqslant R_0-3\delta.$$

In the latter, the inverse function is analytical, as $\frac{\partial\phi}{\partial z}>\frac{1}{2}$ with $|\mathrm{Im}\ z|<R_0-2\delta$. Inequality (16), §4 is thereby proved.

3^o. Estimate $F_1(\phi,\ \epsilon)$. Let $|\mathrm{Im}\ z|<R_0-3\delta$, $|\epsilon|\leq\epsilon_0$. As in view of inequality (16) and conditions 3) and 5) of the basic lemma,

$$|\tilde{F}(z,\varepsilon)+\tilde{\Phi}(z,\varepsilon,\Delta_0^*(\varepsilon))|<\delta,$$

the imaginary parts z_1 and z_{11} (see 10 and 12) do not exceed $R_0-2\delta$. Applying lemma 5, §3 we find, on the basis of (17) and (18), that with $|\mathrm{Im}\ z|<R_0-3\delta$, $|\epsilon|\leq\epsilon_0$

$$|\hat{F}(z,\varepsilon)|\leqslant\frac{4C^2}{\delta^5}. \tag{19}$$

It should be noted that the appearance of C^2 in this inequality is the most substantial element of the proof of theorem 2.

With $|\mathrm{Im}\ \phi|<R_0-4\delta$ and $|\epsilon|\leq\epsilon_0$ we have, in view of 2^o:

$$|\mathrm{Im}\, z(\varphi,\varepsilon)|<R_0-3\delta,$$

and estimate (14) §4, therefore, follows from (19), in view of the definition of $F_1(\phi,\ \epsilon)$ and inequality (10) §4.

4^o. Estimate $|\Delta(\Delta_1,\ \epsilon)-\Delta_0^*(\epsilon)|$. Equation

$$\Delta=\Delta_1-\bar{F}(\varepsilon)-\bar{\Phi}(\varepsilon,\Delta),$$

defining $\Delta(\Delta_1,\ \epsilon)$, belongs to the type discussed in lemma 6, §3. We have seen (see 16) that $|\Delta_0^*(\epsilon)| < 2C$, from which it follows on the basis of formula (11) §4 that: /45

$$|\Delta_0^*(\varepsilon)| < \frac{\Delta_0}{3}. \qquad (20)$$

Lemma 6 is thus applicable, and with $|\Delta_1| \leq C < \frac{\Delta_0}{6}$, $|\epsilon| \leq \epsilon_0$

$$|\Delta(\Delta_1, \varepsilon) - \Delta_0^*(\varepsilon)| < 2|\Delta_1|. \qquad (21)$$

Comparing (20) and (21), we find that with $|\epsilon| \leq \epsilon_0$, $|\Delta_1| \leq C$

$$|\Delta(\Delta_1, \varepsilon)| < \frac{2}{3}\Delta_0.$$

With $|\epsilon| < \epsilon_0$, $|\Delta| < \frac{2}{3}\Delta_0$, according to the Cauchy formula, we have:

$$\left|\frac{\partial\Phi}{\partial\Delta}\right| < \frac{\delta\Delta_0}{\frac{\Delta_0}{3}} < \frac{1}{2}$$

(see inequality 8, 10, §4). Estimate (17) §4 has been proved, as it is obvious that

$$\left|\frac{\partial\Delta}{\partial\Delta_1}\right| = \left|\frac{1}{1 + \frac{\partial\tilde{\Phi}}{\partial\Delta}}\right| < 2.$$

5°. Estimate $|\Phi_1(\phi, \epsilon, \Delta_1)|$. We shall present the difference $z_{111} - z_1$. In view of formulas (10) and (12), it amounts to

$$\Delta_1 + \tilde{\Phi}(z, \varepsilon, \Delta(\Delta_1, \varepsilon)) - \tilde{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon)).$$

According to lemma 5, §3, with $|\mathrm{Im}\, z| \leq R_0$, $|\epsilon| \leq \epsilon_0$, $|\Delta_1| < \frac{\Delta_0}{6}$

$$|\tilde{\Phi}(z, \varepsilon, \Delta(\Delta_1, \varepsilon)) - \tilde{\Phi}(z, \varepsilon, \Delta_0^*(\varepsilon))| < |\Delta - \Delta_0^*|,$$

as $\left|\frac{\partial\Phi}{\partial\Delta}\right| < 1$. Comparing the resulting inequality with inequality (21), we find

$$|z_{1II} - z_1| < 3|\Delta_1|. \qquad (22)$$

Applying lemma 5 §3 to the right side of (8), on the basis of (22), (18) and inequalities (7) and (10) §4, we find that

$$|\hat{\Phi}_1(z, \varepsilon, \Delta)| < \frac{C}{\delta^5} 3|\Delta_1| < \delta^2|\Delta_1| \qquad (23)$$

assuming that $|\epsilon| \leq \epsilon_0$, $|\Delta_1| < \frac{\Delta_0}{6}$,

$$|\operatorname{Im}(z + \Delta_1 + \tilde{F} + \tilde{\Phi})| \leqslant R_0 - 2\delta.$$

The latter inequality is fulfilled if

$$|\operatorname{Im} z| < R_0 - 6\delta, \quad |\Delta_1| < C, \quad |\varepsilon| < \varepsilon_0.$$

Actually then

$$|\tilde{F} + \tilde{\Phi}| < \delta + 2\delta\Delta_0 < 3\delta$$

(see formulas 7, 8 and 17, §4 and inequality 20) in both terms z_{111} and z_1. With $|\operatorname{Im} \phi| \leq R_0 - 7\delta$, $|\Delta_1| < C$ we get, in view of 2°:

$$|\operatorname{Im} z| < R_0 - 6\delta.$$

Estimate (15), §4 therefore follows from (23).

The basic lemma has been proved.

§6. Proof of Theorem 2. /46

6.1. Construction $z(\phi, \epsilon)$ and $\Delta(\epsilon)$. We shall assume that in the basic lemma $\Phi = 0$, and use function $F(z, \epsilon)$ of theorem 2 for $F(z, \epsilon)$. We shall select $\delta_1 > 0$ so that

1) $\sum_{n=1}^{\infty} \delta_n < \frac{R_0}{8}$, где $\delta_n = \delta_{n-1}^{1\frac{1}{2}}$ $(n = 2, 3, \ldots)$;

2) $\delta_1 < \frac{K}{64}$, $\delta_1 < \frac{1}{36}$.

Let $6\delta_1^{12} < \Delta_0 < 1$, $R = R_0$, K--the same as in the theorem. Let $L\epsilon' < C_1 = \delta_1^{12}$, $0 < \epsilon' < \epsilon_0$, C_1 and δ_1 be, respectively, ϵ_0, C and δ of the basic lemma. Then all its assumptions are fulfilled, and with $|\operatorname{Im} \phi_1| \leq R - 7\delta_1$, $|\epsilon| \leq \epsilon'$, $|\Delta_1| \leq C_1$, we get:

$$\varphi_1 \to \varphi_1 + 2\pi\mu + \Delta_1 + F_1(\varphi_1, \varepsilon) + \Phi_1(\varphi_1, \varepsilon, \Delta_1),$$

where

$$|F_1(\varphi_1, \varepsilon)| \leqslant \delta_1^{18} = \delta_2^{12}, \tag{1}$$

$$|\Phi_1(\varphi_1, \varepsilon, \Delta_1)| \leqslant \delta_1^{8} |\Delta_1| < \delta_2 |\Delta_1|, \tag{2}$$

$$|z(\varphi_1, \varepsilon) - \varphi_1| \leqslant \delta_1, \quad \left|\frac{\partial z}{\partial \varphi_1}\right| < 2, \tag{3}$$

$$|\Delta(\Delta_1, \varepsilon)| \leqslant \Delta_0, \tag{4}$$

$$\left|\frac{\partial \Delta}{\partial \Delta_1}\right| < 2.$$

Generally, if we determine the functions $(k = 1, 2, \ldots, n)$

$$\Delta_{k-1}(\Delta_k, \varepsilon), \quad F_k(\varphi_k, \varepsilon), \quad \Phi_k(\varphi_k, \varepsilon, \Delta_k), \quad \varphi_{k-1}(\varphi_k, \varepsilon)^*,$$
$$A_k(\varphi_k, \varepsilon, \Delta_k),$$

that satisfy the conclusion of the basic lemma by substituting ϕ^*_{k-1} for z, ϕ_k for ϕ, R_{k-1} for R_0, $R_k = R_{k-1} - 7\delta_k$ for $R_0 - 7\delta$, δ^*_{k-1} for Δ_0, A_{k-1} for A_0, A_k for A_1, δ_k for δ and $C_k = \delta_k^{12}$ for C with each $k = 1, 2, \ldots, n$, we can introduce functions ϕ_{n+1} and Δ_{n+1} so that the conclusion of the basic lemma may be consistent with them when $k = 1, \ldots, n + 1$. Actually, the inequalities (9) and (10) of §4 will be fulfilled for δ_n in view of the definition δ_1, (11) follows from the inequality $C_{k+1} = C_k^{\frac{1}{1}\frac{1}{2}} < \frac{1}{6} C_k$, and all the other conditions of the lemma will be included in the conclusion (for the functions of the preceding number, of course). We must therefore consider all the above-mentioned functions as having been constructed. The functions $\phi_{n-1}(\phi_n, \epsilon)$, $\Delta_{n-1}(\Delta_n, \epsilon)$ $(n = N, N - 1, \ldots, 1)$ determined the functions

$$z^{(N)}(\varphi_N, \varepsilon) = z(\varphi_1(\ldots(\varphi_N, \varepsilon)\ldots), \varepsilon), \tag{6}$$

$$\Delta_0^{(N)}(\Delta_N, \varepsilon) = \Delta(\Delta_1(\ldots(\Delta_N, \varepsilon)\ldots), \varepsilon). \tag{7}$$

Let us assume that $\Delta_N = 0$, and let $\Delta_0^{(N)}(0, \epsilon) = \Delta^{(N)}(\epsilon)$. Then

$$\Delta(\varepsilon) = \lim_{N\to\infty} \Delta^{(N)}(\varepsilon),$$

$$z(\varphi, \varepsilon) = \lim_{N\to\infty} z^{(N)}(\varphi, \varepsilon).$$ /47

To justify the convergence of $\Delta^{(N)}(\epsilon)$ and $z^{(N)}(\phi, \epsilon)$, we shall point out first of all that, according to definition δ_n, with $\omega > 0$

$$\lim_{N\to\infty} 2^N \delta_N^{\omega} = 0.$$

*ϕ_0 means z, C_0 means Δ_0; $\Delta_{1-1}(\Delta_1, \epsilon) = \Delta(\Delta_1, \epsilon)$.

6.2. Convergence $\Delta^{(N)}(\epsilon)$. The functions $\Delta_0^{(N)}(\Delta_N, \epsilon)$, according to formula (7) and inequality (17) §4, are determined with $|\epsilon| \leq \epsilon_0$, $|\Delta_N| \leq \delta_N^{12}$. As

$$\frac{\partial\Delta_0^{(N)}}{\partial\Delta_N} = \frac{\partial\Delta}{\partial\Delta_1} \cdots \frac{\partial\Delta_{N-1}}{\partial\Delta_N},$$

the following inequality is fulfilled in the mentioned region on the basis of (5);

$$\left|\frac{\partial\Delta_0^{(N)}}{\partial\Delta_N}\right| < 2^N,$$

and as

$$|\Delta_N\ [\Delta_{N+1}(\ldots(\Delta_M, \varepsilon)\ldots\varepsilon), \varepsilon]| \leqslant \delta_N^{12},$$

if $|\Delta_M| \leq \delta_M^{12}$ $(M \geq N)$ then, according to lemma 5, §3

$$|\Delta_0^N\ [\Delta_N(\Delta_{N+1}\ldots(\Delta_M, \varepsilon)\ldots, \varepsilon), \varepsilon] - \Delta_0^{(N)}(0, \varepsilon)| < 2^N \delta_N^{12}.$$

Hence, in view of (7), we conclude that:

$$|\Delta^{(N)}(\varepsilon) - \Delta^{(M)}(\varepsilon)| < 2^N \delta_N^{12},$$

the immediate result of this is an even convergence of $\Delta^{(N)}(\epsilon)$, with $|\epsilon| \leq \epsilon_0$, which also means that $\Delta(\epsilon)$ is analytical.

6.3. Convergence $\Delta^{(N)}(\phi, \epsilon)$. According to the basic lemma, the functions $\phi_{n-1}(\phi_n, \epsilon)$ have been determined with $|\mathrm{Im}\ \phi_n| \leq R_n$, $|\epsilon| \leq \epsilon_0$ and, in view of (3), differ from their argument ϕ by less than δ_n, and therefore

$$|\mathrm{Im}\ \varphi_{n-1}(\varphi_n, \varepsilon)| < R_{n-1}.$$

Thus, formula (6) defines $z^{(N)}(\phi, \epsilon)$ in the band

$$|\mathrm{Im}\ \varphi| \leqslant R_n = R_0 - 7\sum_{k=1}^{n}\delta_k.$$

1

According to condition 1) of selection δ_1, all these bands contain $|\mathrm{Im}\ \phi| \leq \frac{R}{8}$, so that all the functions $z^{(N)}(\phi, \epsilon)$ are defined in the latter.

As

$$|\varphi_N(\varphi_{N+1}\dots(\varphi_M, \varepsilon), \dots, \varepsilon) - \varphi_M| < \sum_{k=N}^{M} \delta_k,$$

and this sum, according to definition δ_n, is not larger than $2\delta_N$, we find the following from (6):

$$|z^{(N)}(\varphi, \varepsilon) - z^{(M)}(\varphi, \varepsilon)| < \left|\frac{\partial z^{(N)}}{\partial \varphi}\right| 2\delta_N.$$

On the basis of (3), /48

$$\left|\frac{\partial z^{(N)}}{\partial \varphi}\right| < 2^N,$$

consequently,

$$|z^{(N)}(\varphi, \varepsilon) - z^{(M)}(\varphi, \varepsilon)| < 2^{N+1}\delta_N,$$

which proves the even convergence of $Z^{(N)}(\phi, \epsilon)$, with $|\mathrm{Im}\ \phi| \le \frac{R}{8}$, $|\epsilon| \le \epsilon_0$.

6.4. We shall define $\phi(z, \epsilon)$ as an inverse function of $z(\phi, \epsilon)$. Since $\delta_n \to 0$ with $n \to \infty$, it follows from inequalities (1) and (2) that

$$\varphi(z, \varepsilon) \to \varphi(z, \varepsilon) + 2\pi\mu,$$

when $z \to A(z, \epsilon, \Delta(\epsilon))$. Theorem 2 has been proved.

§7. Monogenetic Functions

7.1. The concept of monogenesis. In our investigation of the relationship between the solution of equation (1) §2 and parameter μ, we came across an analytic function in the upper and lower semiplanes which was everywhere discontinuous on a real axis. The same characteristics are inherent in all the functions constructed in §6 (see §8)--Δ_n, g_n, ϕ_n, F_n, Φ_n--which are considered as functions of μ. These functions are of a type referred by Borel [9] as monogenetic.

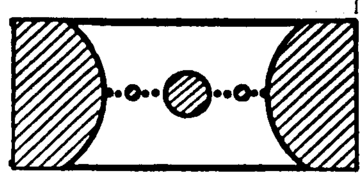

Fig. 2

The Borel monogenetic functions are defined by the set $E = \bigcup_{k=1}^{\omega} E_k$, where $E_k \subseteq E_{k+1}$ represent perfect compact subsets of a complex plane. In this case, E_k represents set M_K^R of points μ of a rectangle on a complex plane $|\text{Im}\,\mu| \leq R$, $0 \leq \text{Re}\,\mu \leq 1$, for which

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{|n|^3} \quad \left(K = \frac{1}{k}\right),$$

that is, a set formed by the elimination from rectangle $|\text{Im}\,\mu| \leq R$, $0 \leq \text{Re}\,\mu \leq 1$ of the circles crosshatched in Fig. 2, $C_{\frac{m}{n},K}$ and radii $\frac{K}{|n|^3}$ with their centers in the rational points $\frac{m}{n}$.

Definition. Function $f(\mu)$ is uniformly differentiable by a perfect compact F of a complex plane, and function $g(\mu)$ is its derivative, if for any $\epsilon \leq 0$ there exists $\delta(\epsilon)$ such that

$$\left|\frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} - g(\mu_3)\right| < \varepsilon,$$

as soon as

$$|\mu_1 - \mu_3| < \delta, \ |\mu_2 - \mu_3| < \delta, \ \mu_1, \mu_2, \mu_3 \in F.$$

The function is monogenetic by $E = \bigcup_{k=1}^{\infty} E_k$ if it is uniformly differentiable by each E_k. /49

In particular, a function uniformly differentiable by E is monogenetic by $E = \bigcup_{k=1} E_k$ and, conversely, a function monogenetic by by $E = \bigcup_{k=1} E_k$ is uniformly differentiable by E. We shall call these functions monogenetic by E, to distinguish them from those monogenetic by $E = \bigcup_{k=1}^{\infty} E_k$.

1) The continuity of the derivative by E_k follows from the monogenetic nature of $E = \bigcup_{k=1}^{\infty} E_k$.

2) If Γ is a linearized curve connecting the two points α and β in E_k, then

$$\int_{\Gamma} f'(\mu)\, d\mu = f(\beta) - f(\alpha).$$

3) A function which is analytic in the domain of each point of a set is monogenetic on it.

4) If E_k contains a region, the $E = \bigcup_{k=1}^{\infty} E_k$ monogenetic function is analytical in it.

An example of a nonanalytic monogenetic function is cited in §2 and proved in point 7.4 (see lemma 10; it is up to the reader to prove that $g(\mu)$ is not analytical with $\mathrm{Im}\,\mu = 0$).

The properties of a monogenetic function may depend largely on its definition of $E = \bigcup_{k=1}^{\infty} E_k$ and the expansion of E into E_k. If the velocity of the decreasing supplementary components of E_k is high enough, then, as Borel showed, the instantaneous functions $E = \bigcup_{k=1}^{\infty} E_k$ possess numerous characteristics of analytic functions (the Cauchy integral, continuous differentiability, uniqueness of instantaneous continuation). The question as to which of these properties is retained in this case will be left aside, as only the definition of a uniform differentiability is used hereafter (§8 and §11).

The class of monogenetic $E = \bigcup_{k=1}^{\infty} E_k$ functions depends not only on E but also on E_k. However, if E results from a different system of sets, $E = \bigcup_{k=1}^{\infty} F_k$, $F_k \subseteq F_{k+1}$, such that

$$E_{\alpha k} \subseteq F_k \subseteq E_{\beta k} \quad (\alpha < 1 < \beta),$$

then the classes of the monogenetic functions $E = \bigcup_{k=1}^{\infty} E_k$ and $E = \bigcup_{k=1}^{\infty} F_k$, coincide. The M_K^R sets (Fig. 2) are inconvenient for the investigation of monogenetic functions in view of the confused nature of the intersections of circles $C_{\frac{m}{n}, K}$. Making use of the preceding remark, we shall /50 replace these sets by another system of sets, N_K^R, so that

1. $M_{2K}^R \subseteq N_K^R \subseteq M_{\frac{K}{2}}^R$.

2. The N_K^R set results from the rectangle $|\mathrm{Im}\,\mu| \le R$, $\mathrm{Re}\,\mu \in (0,1)$ by the elimination of the nonintersecting open circles.

The construction of N_K^R $(K < \frac{1}{9})$ is outlined in point 7.2; it is unwieldy and may be omitted by the reader.

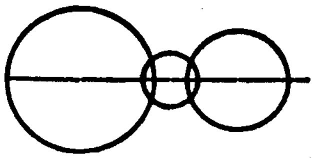

Fig. 3

7.2. The construction of N_K^R. The transformation of M_K^R and N_K^R consists of two operations. The eliminated circles $C_{\frac{m}{n},K}$ are first reduced to circles $C'_{\frac{m}{n},K}$ so that the system $C'_{\frac{m}{n},K}$ ($m = 0,1,\ldots$; $n = 1, 2,\ldots$) contains no "bridges" (see Fig. 3), that is, sets of three circles in which the smaller circle is intersected by two larger ones while the latter two do not intersect. The C' circles are then increased to $C''_{\frac{m}{n},K}$ so that two such circles do not intersect, or that one lies within the other. In this case, the following is to be done

$$C_{\frac{m}{n},K} \supseteq C'_{\frac{m}{n},K} \supseteq C_{\frac{m}{n},\frac{K}{2}},$$
$$C'_{\frac{m}{n},K} \subseteq C''_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},2K}.$$

Then

$$C_{\frac{m}{n},\frac{K}{2}} \subseteq C''_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},2K}$$

and the elimination of the circles $C''_{\frac{m}{n},K}$ from the rectangle will leave the set N_K^R which possesses both of the required characteristics.

LEMMA 7. Let the circles $C_{\frac{m}{n},K}$ and $C_{\frac{p}{q},K}$ $(n \geq q)$ intersect and $K < \frac{1}{9}$. Then $n > 2\sqrt[3]{q^4}$, that is, the smaller circle is much smaller than the larger one.

Proof. Actually, the sum of the radii of the circles is greater than the distance between their centers, so that

$$\frac{K}{n^3} + \frac{K}{q^3} > \left|\frac{p}{q} - \frac{m}{n}\right|.$$

As $pn - qm \neq 0$, then $\left|\frac{p}{q} - \frac{m}{n}\right| \geqslant \frac{1}{qn}$, и

$$K(n^3 + q^3) \geqslant q^2 n^2;$$

in view of the inequality $n \geq q$, we get: /51

$$K(n^3+q^3) \geqslant q^4,$$

or

$$n^3 > \frac{q^4}{K} - q^3.$$

Bearing in mind that $K \frac{1}{9}$, we find that:

$$n^3 > 9q^4 - q^3 \geqslant 8q^4,$$

which completes the proof.

Operation 1 -- construction of $C'_{\frac{m}{n}, K}$. This construction consists of an infinite number of successive stages, so that the circles $C'_{\frac{m}{n}, K}$ $(0 \leq m \leq n)$, found to be constructed after the n-th stage, possess the following properties:

A_n. No circle $C_{\frac{m_1}{n_1}, K}$ $(n_1 > n)$ can connect circle $C'_{\frac{m}{n}, K}$ with circle $C'_{\frac{m_2}{n_2}, K}$ $(n_2 \leq n)$ if these circles, $C'_{\frac{m}{n}, K}$ and $C'_{\frac{m_2}{n_2}, K}$, do not intersect each other.

B_n. $$C_{\frac{m}{n}, \frac{K}{2}} \subseteq C'_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, K}.$$

We shall begin with the first stage. Let $C'_{\frac{m}{1}, K} = C_{\frac{m}{1}, K}$. Let property B_1 be fulfilled. The property A_1 is also fulfilled, as the diameter of the circle $C_{\frac{m_1}{n_1}, K}$ $(n_1 > n)$ is smaller than

$$\frac{2K}{n_1^3} < \frac{2}{9 \cdot 8} \quad \left(K < \frac{1}{9}\right),$$

and the distance between the circles $C_{\frac{0}{1}, K}$ and $C_{\frac{1}{1}, K}$ is greater than

$$1 - 2K > \frac{2}{3}.$$

The first stage has been completed.

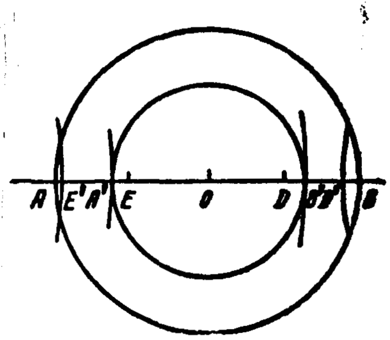

Fig. 4.

Let us assume that stage n-1 was consistently carried out. We shall examine any circle $C = C_{\frac{m}{n}, K}$ (Fig. 4). Let O be its center, AB the diameter lying on a real axis, and E and D the middle parts of AO and OB. Circle C can be intersected only by the circles $C'_{\frac{m_2}{n_2}, K}$ $(n_2 < n)$ where $C_{\frac{m_2}{n_2}, K}$ intersects with C (in view of the property B_k, $k \leq n-1$). Further, all such circles $C'_{\frac{m_2}{n_2}, K}$ intersect each other (in view of the property A_k, $k \leq n-1$).

Let us arrange the circles in a decreasing order n_2 (increase of circles): /52

$$C_i = C_{\frac{m_{2,i}}{n_{2,i}}, K} \quad (n = n_{2,0} > n_{2,1} > \ldots > n_{2,l} \geqslant 1).$$

On the basis of lemma 7, $n_{2,i} \geq 2n_{2,i+1}$ $(0 \leq i \leq l-1)$, from which it follows that $n > 2^l$ or $l < \log_2 n$. Thus, at the intersection with the AB diameter, the circumferences of circles $C_{\frac{m_2}{n_2}, K}$ produce no more than 2 $\log_2 n$ points. Therefore, the parts into which these points divide the segments BD and AE will include parts which are not longer than $\frac{K}{4n^3 \log_2 n}$.

But the diameter of circle $C_{\frac{m_1}{n_1}, K}$ $(n_1 > n)$ intersecting with C, according to lemma 7, does not exceed

$$\frac{K}{8n^4} < \frac{K}{4n^3 \log_2 n}.$$

We shall take ends B' and A' of the larger parts BD and AE, which are near to O and designated as B'D' and A'E, for the ends of diameter $C'_{\frac{m}{n}, K}$. This selection does not contradict the characteristic B_n. It is clear that if the circumference $C_1 = C_{\frac{m_1}{n_1}, K}$ $(n_1 > n)$ intersects with $C'_{\frac{m}{n}, K}$, it lies within C, and can only intersect with C_i of the circles $C_{\frac{m_2}{n_2}, K}$ $(n_2 \leq n)$. But as the diameter of C_1 is shorter than B'D' and A'E', C_1

can intersect only the C_i which intersects with $C'_{\frac{m}{n},K}$. Property A_n is therefore also fulfilled, and the method of completing the n-th stage is thereby indicated.

The completion of all the stages will produce a system of circles $C'_{\frac{m}{n},K}$ possessing the following properties:

A. No circle $C'_{\frac{m_1}{n_1},K}$ can connect $C'_{\frac{m_2}{n_2},K}$ with $C'_{\frac{m_3}{n_3},K}$ if

$$n_1 > n_2,\ n_1 > n_3 \ u\ C'_{\frac{m_2}{n_2},K} \cap C'_{\frac{m_3}{n_3},K} = 0.$$

B. $C_{\frac{m}{n},\frac{K}{2}} \subseteq C'_{\frac{m}{n},K} \subseteq C_{\frac{m}{n},K}$.

Property B follows from B_n, and property A from A_{n_2}, if $n_2 \geq n_3$, and from A_{n_3} if $n_3 \geq n_2$.

Operation 2 -- construction of $C'_{\frac{m}{n},K}$. We shall now increase the circles of the system $C'_{\frac{m}{n},K}$.

We shall refer to the totality of $C'_{\frac{m_i}{n_i},K}$ $(n_i > n)$ which can be connected with C by a monotonic finite chain of intersecting circles $C'_{\frac{m_{j_k}}{n_{j_k}},K}$ $(0 \leq k \leq l_k)$ as the "tail" of $C = C'_{\frac{m}{n},K}$: /53

$$\frac{m_{j_0}}{n_{j_0}} = \frac{m}{n},\quad n_{j_k} < n_{j_{k+1}},\quad C'_{\frac{m_{j_k}}{n_{j_k}},K} \cap C'_{\frac{m_{j_{k+1}}}{n_{j_{k+1}}},K} \neq 0,\quad \frac{m_{j_{l_i}}}{n_{j_{l_i}}} = \frac{m_i}{n_i}.$$

It is clear that if circle C_1 is included in the tail of circle C_2, then the tail of C_1 is entirely included in the tail of C_2. Moreover, if the tails of C_1 and C_2 intersect,* one of these tails is

* It can be readily seen that if two tails intersect as sets of points, they have a common circle.

entirely included in the other. We shall prove this. Let us assume the opposite: let it be possible to connect circles C_1 and C_2 with a general circle of their tails, C_3, by monotonic chains. Two such chains together connect C_1 and C_2. Of the chains connecting C_1 and C_2 we shall select one consisting of the smallest number of circles. Intersecting within it are only the neighboring circles (see Fig. 5; in the above system of circles the tail of the largest one is crosshatched). If this chain is monotonic, our assertion has been proved. If the chain is not monotonic, it contains a circle connecting two larger ones which is in contradiction with property A of operation 1. Thus if two tails intersect, one of them contains the other one.

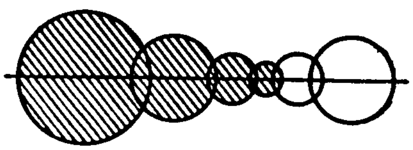

Fig. 5

Let α and β represent the upper and lower boundaries of the points of a real axis covered by the tail of circle $C = C'_{\frac{m}{n}, K}$. The circle with the $\alpha\beta$ diameter will then be circle $C''_{\frac{m}{n}, K}$. It follows from the above that the circumferences of two such circles do not intersect.* It is obvious that $C''_{\frac{m}{n}, K} \supseteq C'_{\frac{m}{n}, K}$. We will show that

$$C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 3K}.$$

Indeed, on the basis of lemma 7, it is easy to estimate the dimension of the tail C. Let circle C_1 be included in the tail of C, and the monotonic chain connecting C_1 with C consist of N circles. As each of the circles, according to lemma 7, is at least 8 times smaller than the preceding one, the sum total of their diameters does not exceed $\frac{1}{7}$ of the C diameter in any N. Hence, it follows that α and β are removed from $C'_{\frac{m}{n}, K}$ by not more than $\frac{1}{7}$ of the diameter of $C_{\frac{m}{n}, K}$, and by not more than $1-\frac{2}{7}$ of the $C_{\frac{m}{n}, K}$ radius, hence

$$C''_{\frac{m}{n}, K} \subseteq C_{\frac{m}{n}, 3K}.$$

The construction of N_K^R has been completed. /54

*But they may be contiguous.

7.3. Differentiation of a sequence. The extension into the complex plane μ was undertaken primarily in connection with the following lemma which does not hold true if by the set N_K^R we mean its part lying on a real axis.

LEMMA 8. Let the sequence of functions $f_n(\mu)$, which are monogenetic on set N_K^R, converge on it in proportion to $f(\mu)$, and the derivatives converge in proportion to $g(\mu)$. Then $f(\mu)$ is monogenetic on N_K^R and $f'(\mu) = g(\mu)$.

Proof. 1. Let $\epsilon > 0$. We must find $\delta > 0$ such that

$$\left|\frac{f(\mu_1)-f(\mu_2)}{\mu_1-\mu_2} - g(\mu_3)\right| < \varepsilon,$$

when

$$|\mu_1-\mu_3| < \delta, \quad |\mu_2-\mu_3| < \delta, \quad \mu_1, \quad \mu_2, \mu_3 \in N_K^R.$$

If $\delta > 0$ is sufficiently small, then all these points lie within a single component of N_K^R.

We will show that in this case the points μ_1 and μ_2 can be connected within N_K^R by a linearized curve Γ in such a way as to fulfill the following conditions:

1) for any point $\mu \in \Gamma$ $|\mu-\mu_3| < 2\delta$;

2) the length of Γ is smaller than $2|\mu_1-\mu_2|$.

Indeed, we shall connect the points μ_1 and μ_2 with the segment $\mu_1 \mu_2$ (see Fig. 6). This segment may intersect with some circles C_i, whose elimination from rectangle $|\text{Im}\ \mu| \leq R$, $\text{Re}\ \mu \in [0,1]$ resulted in the formation of set N_K^R. These circles do not intersect in pairs and do not separate μ_1 from μ_2 in N_K^R as the points μ_1 and μ_2 lie in a single component.

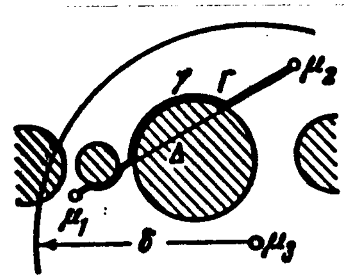

Fig. 6

The circles C_i form nonintersecting intervals Δ_i on μ_1 and μ_2. For each of such intervals Δ_i we shall substitute arc γ_i, the smaller of the two arcs into which $\mu_1 \mu_2$ divide the circle C_i.

This substitution will increase the length of Δ_i not more than $\frac{\pi}{2}$ times, and the length of Γ will therefore be less than $2|\mu_1-\mu_2|$. The distance $|\mu_1-\mu_2|$, according to the condition, does not exceed 2δ, and all the points γ_i are therefore removed from the center of Δ_i by less than δ. The latter point, like all the points of segment $\mu_1\mu_2$, lies in circle $|\mu-\mu_3|<\delta$, and for any point $\mu\in\gamma_i$, therefore

$$|\mu-\mu_3|<2\delta.$$

Thus curve Γ is the one we searched for.

2. We have already noted that if $\phi(\mu)$ is monogenetic in N_K^R and Γ is a linearized curve with ends μ_1 and μ_2, then

$$\int_\Gamma \varphi'(\mu)\,d\mu=\varphi(\mu_2)-\varphi(\mu_1).$$

(This can be proved by merely comparing the integral with the integral sum.)

Applying this equality to the above curve Γ and monogenetic functions $f_n(\mu)$, we get: /55

$$\int_\Gamma f_n'(\mu)\,d\mu=f_n(\mu_2)-f_n(\mu_1).$$

In view of the uniform convergence of f_n and f, f_n' and g, it is possible to proceed to the following limit on the right and left sides:

$$\int_\Gamma g(\mu)\,d\mu=f(\mu_2)-f(\mu_1).$$

3. We shall estimate

$$\left|\frac{f(\mu_2)-f(\mu_1)}{\mu_2-\mu_1}-g(\mu_3)\right|.$$

In this connection we shall examine the integral

$$\int_\Gamma (g(\mu)-g(\mu_3))\,d\mu=f(\mu_2)-f(\mu_1)-(\mu_2-\mu_1)\,g(\mu_3).$$

We have:

$$\left|\int_{\Gamma}(g(\mu)-g(\mu_3))\,d\mu\right|\leqslant\int_{\Gamma}|g(\mu)-g(\mu_3)||d\mu|\leqslant\max_{\mu\in\Gamma}|g(\mu)-g(\mu_3)|\cdot 2|\mu_2-\mu_1|,$$

as the length of Γ is less than $2|\mu_2-\mu_1|$.

Thus

$$\left|\frac{f(\mu_2)-f(\mu_1)}{\mu_2-\mu_1}-g(\mu_3)\right|\leqslant 2\max_{\mu\in\Gamma}|g(\mu)-g(\mu_3)|.$$

The right side of the last inequality, according to property 1) of curve Γ, is a (double) increment of $g(\mu)$ on a line segment shorter than 2δ and, in view of the continuity of the continuous function $g(\mu)$ on the N_K^R compact, it tends to become zero along with δ. Lemma 8 has been proved.

7.4. The functions of several variables and their uses. Eventually we will need functions which are analytic for some variables and monogenetic for others.

Let z be an angular variable (changing within the Im $z \in (ab)$ region*) and have a 2π period,** the variables ϵ and Δ change in the neighborhood of zero, and $\mu \in N_K^R$.

Definition. Function $f(z, \epsilon, \Delta, \mu)$ is analytic for z, ϵ, Δ and monogenetic for $\mu \in N_K^R$ if the sequence

$$f(z, \varepsilon, \Delta, \mu)=\sum f_{kmn}(\mu)\,e^{ikz}\varepsilon^m\Delta^n,$$

in which the coefficients are monogenetic functions of $\mu \in N_K^R$, converges uniformly along with the derivative to μ, with $\mu \in N_K^R$ and z, ϵ, Δ changing in the mentioned regions.

It is obvious that such a function is continuous, and

a) with μ fixed, it is analytic for z, ϵ, Δ, and

b) with z, ϵ, Δ fixed, it is monogenetic for $\mu \in N_K^R$. Property b) follows from lemma 8.

* The boundaries may depend on μ.

** That is, when increased by a 2π function, z gets an increment of 0 or 2π.

LEMMA 9. Let functions $h_i(z, \epsilon, \Delta, \mu)$ be monogenetic for $\mu \in E$ and analytic for z, ϵ, Δ. The following will then possess the same property in the corresponding regions: /56

1) functions

$$h_1(z, \varepsilon, \Delta, \mu) + h_2(z, \varepsilon, \Delta, \mu), \quad h_1(z, \varepsilon, \Delta, \mu) h_2(z, \varepsilon, \Delta, \mu),$$
$$h_1(h_2(z, \varepsilon, \Delta, \mu), \varepsilon, \Delta, \mu), \quad h_1(z, \varepsilon, h_2(z, \varepsilon, \Delta, \mu), \mu);$$

2) solution $\phi(z, \epsilon, \Delta, \mu)$ of equation $h(\phi, \epsilon, \Delta, \mu) = z$;

3) solution $\gamma(z, \epsilon, \Delta, \mu)$ of equation $h(z, \epsilon, \gamma, \mu) = \Delta$;

4) partial derivatives h for z, ϵ, Δ;

5) integral for parameter $\int_0^{2\pi} h(z, \epsilon, \Delta, \mu)\,dz$,

and the usual rules of differentiation apply in all of these cases; for example in case 2)

$$\frac{\partial \varphi}{\partial \mu} = -\frac{\frac{\partial h}{\partial \mu}}{\frac{\partial h}{\partial \varphi}}.$$

The proof reiterates the reasoning well known form the usual analysis, and is then omitted.

LEMMA 10. Let function $f(z, \epsilon, \Delta, \mu) = \bar{f}$ be analytic for z in the region $|\mathrm{Im}\, z| \leq R$; ϵ, $|\epsilon| \leq \epsilon_0$; $|\Delta| \leq \Delta_0$ and monogenetic for $\mu \in N_K^R$, and let ϵ of the mentioned region be

$$|f| \leqslant C, \quad \left|\frac{\partial f}{\partial \mu}\right| \leqslant L.$$

Then the solution to the equation

$$g(z + 2\pi\mu, \varepsilon, \Delta, \mu) - g(z, \varepsilon, \Delta, \mu) = f(z, \varepsilon, \Delta, \mu)$$

is monogenetic for $\mu \in N_K^R$ and analytic for z in the region $|\mathrm{Im}\,(z - 2\pi\mu)| \leq R - 2\delta$, ϵ, $|\epsilon| \leq \epsilon_0$, Δ, $|\Delta| \leq \Delta_0$, and in this region

$$|g| \leqslant \frac{4C}{K\delta^3}, \quad \left|\frac{\partial g}{\partial z}\right| \leqslant \frac{8C}{K\delta^4}, \quad \left|\frac{\partial^2 g}{\partial z^2}\right| \leqslant \frac{10C}{K\delta^5},$$
$$\left|\frac{\partial g}{\partial \mu}\right| \leqslant \frac{C+L}{K^2}\frac{10^3}{\delta^6}, \quad \left|\frac{\partial^2 g}{\partial z \partial \mu}\right| \leqslant \frac{C+L}{K^2}\frac{10^3}{\delta^7}.$$

Proof. The solution is given with μ fixed by series (2) §2

$$\sum_{n\neq 0} \frac{f_n(\mu, \varepsilon, \Delta)}{e^{2\pi i n\mu} - 1} e^{inz},$$

whose uniform convergence at $|\mathrm{Im}\,(z - 2\pi\mu)| \leq R - 2\delta$ is to be established, because

$$f_n(\mu, \varepsilon, \Delta) = \sum f_{nkl}(\mu)\, \varepsilon^k \Delta^l.$$

But the uniform convergence of this series was established in §2, along with the unknown estimates of g and $\frac{\partial g}{\partial z}$ in the proof of theorem 1', as

$$N_K^{\frac{1}{2\pi}} \subseteq M_{\frac{K}{2}}^{\frac{1}{2\pi}}.$$

The estimates of the other derivatives are found by differentiating the series according to the usual formulas, taking inequality (13) §2 into consideration. /57

§8. The Functional Relation of Theorem 2 to μ.

8.1. We have seen in 7.4 that there is a monogenetic relationship between the solution of the linear equation (1) §2 and μ. The monogenetic relationship between μ and the functions Δ_n, F_n, Φ_n, g_n, $\Delta^{(n)}$ constructed in §6 will be proved in this paragraph.

It appears that the region of monogenetic relations grows narrower (by $|\mathrm{Im}\, 2\pi\mu|$ on each step) as n increases, and the author was unable to establish any monogenetic relationship between μ and the solution of equation (1) §4.

The monogenetic relationship between $\Delta^{(n)}$ and real μ is used in §11. There we shall rely also on the (uniform for n) small value $\frac{\partial \Delta^{(n)}}{\partial \mu}$ with small ϵ.

To abbreviate the cumbrous expressions in this paragraph, argument ϵ is omitted from all functions, just as the dependence on μ has been ignored before, and only z, ϕ, ϵ, Δ considered as arguments.

The construction of $\Delta^{(n)}(\mu)$ in §6 looks like the following.

Such new parameters as $\phi = \phi_n(\phi_{n-1}, \mu)$ and $\Delta_{n-1} = \Delta_{n-1}(\Delta_n, \mu)$ were added step by step so that the transformation

$$\varphi_{n-1} \to \varphi_{n-1} + 2\pi\mu + \Delta_{n-1}(\Delta_n, \mu) + F_{n-1}(\varphi_{n-1}, \mu) + \Phi_{n-1}(\varphi_{n-1}, \Delta_{n-1}(\Delta_n, \mu)\mu)$$

was converted to the following transformation

$$\varphi_n \to \varphi_n + 2\pi\mu + \Delta_n + F_n(\varphi_n, \mu) + \Phi_n(\varphi_n, \Delta_n, \mu)$$

with considerably smaller F and Φ values, and $\phi_0 = z$, $F_0^i = F$, $\Phi_0 = 0$, $\Delta_0 = \Delta$.

Further, the construction of $\Delta^{(n)}(\mu)$ was such that the transformation

$$z \to z + 2\pi\mu + \Delta^{(n)}(\mu) + F(z)$$

in the variable ϕ_n became

$$\varphi_n \to \varphi_n + 2\pi\mu + F_n(\varphi_n, \mu) + \Phi_n(\varphi_n, 0, \mu),$$

in which case we supposed

$$\begin{aligned} \Delta_k^{(n)}(\mu) &= \Delta_k(\Delta_{k+1}^{(n)}(\mu), \mu) \quad (k = 0, 1, \dots, n-1), \\ \Delta_n^{(n)}(\mu) &= 0. \end{aligned} \tag{1}$$

We thus obtained the following:

$$\Delta_0^{(n)}(\mu) = \Delta^{(n)}(\mu).$$

THEOREM 3. By the terms of theorem 2, with fairly small $\epsilon > 0$, $0 < K < \frac{1}{9}$ values

$$\Delta(\mu) = \lim_{n \to \infty} \Delta^{(n)}(\mu),$$

where the functions $\Delta^{(n)}(\mu)$ are monogenetic for $\mu \in N_K^{r_n}$ $(r_n > 0)$, and under these conditions $\left| \frac{\partial \Delta^{(n)}}{\partial \mu} \right| < 6L\,|\epsilon|$.

The proof of this theorem rests on the following lemma which repeats the basic lemma (see §§4 and 5).

LEMMA 11. Let a family of analytic representations of a circle be analytically Δ-dependent and monogenetically $\mu \in N_K^r$-dependent /58

$$z \to A_0(z, \Delta, \mu) = z + 2\pi\mu + F(z, \mu) + \Delta + \Phi(z, \Delta, \mu)$$

and the numbers $R_0 > 0$, $\frac{1}{9} > K > 0$, $\delta > 0$, $C > 0$, $0 < \Delta_0 < 1$, $0 < r < \frac{1}{2\pi}$, $2\pi r \leqslant R_0 - 5\delta$ such that

1) $F(z + 2\pi, \mu) = F(z, \mu)$, $\Phi(z + 2\pi, \Delta, \mu) = \Phi(z, \Delta, \mu)$;

2) with $\operatorname{Im} z = \operatorname{Im} \mu = \operatorname{Im} \Delta = 0$ always $\operatorname{Im} F = \operatorname{Im} \Phi = 0$;

3) with $|\operatorname{Im} z| \leq R_0$, $\mu \in N_K^r$, $|\Delta| \leq \Delta_0$

$$|F(z, \mu)| \leqslant C, \tag{2}$$

$$\left|\frac{\partial F(z, \mu)}{\partial \mu}\right| \leqslant C, \tag{3}$$

$$|\Phi(z, \mu, \Delta)| \leqslant \delta^2 |\Delta|, \tag{4}$$

$$\left|\frac{\partial \Phi(z, \mu, \Delta)}{\partial \mu}\right| \leqslant \delta^2 |\Delta|; \tag{5}$$

4) the number δ satisfies inequality

$$\delta < \frac{K^3}{5 \cdot 10^4}; \tag{6}$$

5) $C = \delta^{27}$, $\Delta_0 = \delta^{26}$.

Then there exist functions $z(\phi, \mu)$, $\Delta(\Delta_1, \mu)$, which are analytic for ϕ, Δ_1 and monogenetic for $\mu \in N_K^r$ such that

1. The following is identical

$$z(A_1(\varphi, \mu, \Delta_1), \mu) = A_0(z(\varphi, \mu), \Delta(\Delta_1, \mu), \mu),$$

where

$$A_1(\varphi, \mu, \Delta_1) \equiv \varphi + 2\pi\mu + \Delta_1 + F_1(\varphi, \mu) + \Phi_1(\varphi, \mu, \Delta_1).$$

2. $F_1(\varphi + 2\pi, \mu) = F_1(\varphi, \mu)$, $\Phi_1(\varphi + 2\pi, \mu, \Delta_1) = \Phi_1(\varphi, \mu, \Delta_1)$,
$$z(\varphi + 2\pi, \mu) = z(\varphi, \mu) + 2\pi.$$

3. With $\operatorname{Im} \phi = \operatorname{Im} \Delta_1 = \operatorname{Im} \mu = 0$ always $\operatorname{Im} z = \operatorname{Im} \Delta = \operatorname{Im} F_1 = \operatorname{Im} \Phi_1 = 0$

4. With $|\Delta_1| \leq \delta^{28}$, $|\operatorname{Im}\phi| \leq R_0 - 7\delta - |\operatorname{Im} 2\pi\mu|$, $\mu \in N^r_K$, the constructed functions are analytic for ϕ, Δ_1, monogenetic for $\mu \in N^r_K$ and have the following correlations:

$$|F_1| \leqslant \frac{C^2}{\delta^6}, \tag{7}$$

$$|\Phi_1| \leqslant \frac{C}{\delta^6} |\Delta_1|, \tag{8}$$

$$\left|\frac{\partial F_1}{\partial \mu}\right| \leqslant \frac{C^2}{\delta^{13}}, \tag{9}$$

$$\left|\frac{\partial \Phi_1}{\partial \mu}\right| \leqslant \frac{C}{\delta^{13}} |\Delta_1|, \tag{10}$$

$$\left|\frac{\partial z}{\partial \mu}\right| \leqslant \frac{C}{\delta^7}, \tag{11}$$

$$\left|\frac{\partial \Delta}{\partial \mu}\right| \leqslant 4C, \tag{12}$$

$$|\Delta(\Delta_1, \mu)| \leqslant \Delta_0, \tag{13}$$

$$|z(\varphi, \mu) - \varphi| \leqslant \frac{C}{\delta^4}, \tag{14}$$

$$\left|\frac{\partial \Delta}{\partial \Delta_1}\right| \leqslant 2, \tag{15}$$

$$\left|\frac{\partial z}{\partial \varphi}\right| \leqslant 2. \tag{16}$$

8.2. The proof of lemma 11 is more unwieldy than the proof of the basic lemma. The construction reiterates the reasoning in 5.1 with the only difference that μ changes from a fixed real number to an independent complex variable. In the construction of $\Delta(\Delta_1)$, $z(\phi)$, g, F_1 and Φ_1, according to 5.1., use is made of the integration by z, the solution of equation (1) 2, the construction of an inverse function and the substitution of a function in a function. According to lemmas 7.4, all these operations do not extend beyond the class of functions which are monogenetic for $\mu \in N^r_K$ and analytic for z, Δ, ϕ, Δ_1 in the respective domains. /59

Therefore, only inequalities (9), (10), (11), (12), which are not bound in the basic lemma, call for a special examination. Their proof is based on the following estimates.

1°. Estimate $\frac{\partial g^*}{\partial \mu}$. On the basis of 5.1 and 7.4, and in view of the terms of the lemma, when

$$|\operatorname{Im} z| \leqslant R_0, \ \mu \in N^r_K, \ |\Delta| \leqslant \Delta_0$$

$$\left|\frac{\partial \tilde{F}}{\partial \mu}\right| \leqslant 2C, \quad \left|\frac{\partial \tilde{\Phi}}{\partial \mu}\right| \leqslant 2\delta^2 |\Delta_0| \leqslant 2C.$$

Thus in the right-hand side of equation (2) §5 is a derivative of μ, not exceeding 4C. Applying lemma 10, we find:

$$|g^*| \leqslant \frac{16C}{K\delta^3}, \quad (17)$$

$$\left|\frac{\partial g^*}{\partial z}\right| \leqslant \frac{32C}{K\delta^4}, \quad (18)$$

$$\left|\frac{\partial^2 g^*}{\partial z^2}\right| \leqslant \frac{40C}{K\delta^5}, \quad (19)$$

$$\left|\frac{\partial g^*}{\partial \mu}\right| \leqslant \frac{5\cdot 10^9 C}{K^2\delta^6}, \quad (20)$$

$$\left|\frac{\partial^2 g^*}{\partial z \partial \mu}\right| \leqslant \frac{5\cdot 10^9 C}{K^2\delta^7} \quad (21)$$

with $|\mathrm{Im}(z - 2\pi\mu)| \leqslant R - 2\delta,\ \mu \in N_K^r,\ |\Delta| \leqslant \Delta_0.$

2°. Estimate $\frac{\partial \Delta_0^*}{\partial \mu}$. It follows from equation (4) §5 and 7.4 that

$$\frac{\partial \Delta_0^*(\mu)}{\partial \mu} = -\frac{\frac{\partial \overline{F}}{\partial \mu} + \frac{\partial \overline{\Phi}}{\partial \mu}}{1 + \frac{\partial \overline{\Phi}}{\partial \Delta}}.$$

Estimating Δ_0 as in 1°, 5.2, we find:

$$|\Delta_0^*| < 2C < \frac{\Delta_0}{2}.$$

With $|\Delta| \leq \frac{\Delta_0}{2}$ and the Cauchy integral we find from (4):

$$\left|\frac{\partial \Phi}{\partial \Delta}\right| \leqslant \frac{\delta^2 \Delta_0}{\frac{\Delta_0}{2}} = 2\delta < \frac{1}{2}, \quad \left|\frac{\partial \overline{\Phi}}{\partial \Delta}\right| < \frac{1}{2}, \quad \left|\frac{\partial \widetilde{\Phi}}{\partial \Delta}\right| < 1.$$

Consequently, $|1 + \frac{\partial \overline{\Phi}}{\partial \Delta}| > \frac{1}{2}$ at $|\Delta| \leq \frac{\Delta_0}{2}$. On the basis of (3), (5) and /60
lemma 9, therefore

$$\left|\frac{\partial \Delta_0^*}{\partial \mu}\right| < 2(C + \delta^2 \Delta_0).$$

In view of (6), $\delta^2\Delta_0 < C$, so that

$$\left|\frac{\partial\Delta_0^*}{\partial\mu}\right| < 4C \tag{22}$$

at $\mu \in N_N^r$.

3°. Estimate $\frac{\partial g}{\partial \mu}$. According to 7.4 and 5.1,

$$\frac{\partial g}{\partial\mu} = \frac{\partial g^*}{\partial\mu} + \frac{\partial g^*}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}, \tag{23}$$

$$\frac{\partial^2 g}{\partial z\partial\mu} = \frac{\partial^2 g^*}{\partial z\partial\mu} + \frac{\partial^2 g^*}{\partial z\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}. \tag{24}$$

We shall first estimate $\frac{\partial g^*}{\partial\Delta}$ and $\frac{\partial^2 g^*}{\partial z\partial\Delta}$. Note that equation

$$g^*(z + 2\pi\mu,\ \Delta,\ \mu) - g^*(z,\ \Delta,\ \mu) = -\tilde{F}(z,\ \mu) - \tilde{\Phi}(z,\ \Delta,\ \mu)$$

when differentiated by Δ produces equation

$$\frac{\partial g^*}{\partial\Delta}(z + 2\pi\mu,\ \Delta,\ \mu) - \frac{\partial g^*}{\partial\Delta}(z,\ \Delta,\ \mu) = -\frac{\partial\tilde{\Phi}}{\partial\Delta}$$

of the same type in relation to $\frac{\partial g^*}{\partial\Delta}$, and we can make use of lemma 10. To this end we shall estimate $\frac{\partial\tilde{\Phi}}{\partial\Delta}$ by the use of the Cauchy integral: with $|\mathrm{Im}\ z| \le R$, $|\Delta| \le \frac{\Delta_0}{2}$

$$\left|\frac{\partial\tilde{\Phi}}{\partial\Delta}\right| \leqslant \frac{2\delta^2\Delta_0}{\frac{\Delta_0}{2}} < 4\delta^2.$$

According to lemma 10, with $|\mathrm{Im}\ (z - 2\pi\mu)| \le R_0 - 2\delta$, $|\Delta| \le \frac{\Delta_0}{2}$, $\mu \in N_K^r$

$$\left|\frac{\partial g^*}{\partial\Delta}\right| < \frac{4}{K\delta^3}4\delta^2,$$

$$\left|\frac{\partial^2 g^*}{\partial\Delta\,\partial z}\right| < \frac{8}{K\delta^4}4\delta^2.$$

Substituting these estimates, estimates (20), (21) and estimate Δ_0^* from 2° in the formulas (23), (24), we find:

$$\left|\frac{\partial g}{\partial \mu}\right| < \frac{5C}{K^2}\frac{10^3}{\delta^6} + \frac{16}{K\delta}4C < \frac{C10^4}{K^2\delta^6},$$
$$\left|\frac{\partial^2 g}{\partial z \partial \mu}\right| < \frac{5\cdot 10^3 C}{K^3\delta^7} + \frac{32}{K\delta^2}4C < \frac{C10^4}{K^2\delta^7}$$

with $|\text{Im}\,(z - 2\pi\mu)| \leq R - 2\delta$, $\mu \in N_K^r$.

4°. Estimate $\frac{\partial \Delta(\Delta_1, \mu)}{\partial \mu}$. Analogically to 2°, we have:

$$\frac{\partial \Delta}{\partial \mu} = -\frac{\frac{\partial \bar{F}}{\partial \mu} + \frac{\partial \bar{\Phi}}{\partial \mu}}{1 + \frac{\partial \bar{\Phi}}{\partial \Delta}},$$

and if $|\Delta| \leq \frac{\Delta_0}{2}$ then, as in 2°, we get: /61

$$\left|\frac{\partial \Delta}{\partial \mu}\right| < 4C.$$

All that is needed to fulfill the inequality, $|\Delta| < \frac{\Delta_0}{2}$, is $|\Delta_1| \leq \delta^{27}$.

Actually, when (as is shown in §5) $|\Delta_0^*| \leq 2C$, $|\Delta - \Delta_0^*| \leq 2|\Delta_1|$, and as $C = \delta^{27}$, then at $|\Delta_1| \leq \delta^{27}$ we have:

$$|\Delta(\Delta_1, \mu)| \leqslant 4\delta^{27} < \frac{\delta^{26}}{2} = \frac{\Delta_0}{2}.$$

So, at $|\Delta_1| \leq \delta^{27}$, $\mu \in N_K^r$,

$$\left|\frac{\partial \Delta(\Delta_1, \mu)}{\partial \mu}\right| < 4C. \tag{25}$$

At the same time we have shown that with $|\Delta_1| \leq \delta^{27}$, the estimates of point 1° are valid.

5°. Estimate $\frac{\partial \hat{F}_1}{\partial \mu}$. From 5.1 and 7.4 we find:

$$\frac{\partial \hat{F}_1(z,\mu)}{\partial\mu}=\left[\frac{\partial g(z_I,\mu)}{\partial\mu}-\frac{\partial g(z_{II},\mu)}{\partial\mu}\right]+\left[\frac{\partial g(z_I,\mu)}{\partial z}-\frac{\partial g(z_{II},\mu)}{\partial z}\right]2\pi+ + \frac{\partial g(z_I,\mu)}{\partial z}\left[\frac{\partial\widetilde{F}}{\partial\mu}+\frac{\partial\widetilde{\Phi}}{\partial\mu}+\frac{\partial\widetilde{\Phi}}{\partial\Delta}\frac{\partial\Delta_0^{*}}{\partial\mu}\right], \tag{26}$$

where

$$z_I=z+2\pi\mu+\widetilde{F}(z,\mu)+\widetilde{\Phi}(z,\mu,\Delta_0^{*}(\mu)), \tag{27}$$
$$z_{II}=z+2\pi\mu. \tag{28}$$

The first two brackets on the right-hand side of (26) are estimated by the use of a finite increment lemma (lemma 5, §3). We have:

$$\left|\frac{\partial g(z_I)}{\partial\mu}-\frac{\partial g(z_{II})}{\partial\mu}\right|\leqslant|z_I-z_{II}|\left|\frac{\partial^2 g}{\partial\mu\,\partial z}\right|;$$

Substituting for $z_I - z_{II}$ and $\frac{\partial^2 g}{\partial\mu\partial z}$ their estimates, we get:

$$\left|\frac{\partial g(z_I)}{\partial\mu}-\frac{\partial g(z_{II})}{\partial\mu}\right|\leqslant\frac{4\cdot10^4C^2}{K^2\delta^7}$$

and, analogically,

$$\left|\frac{\partial g(z_I)}{\partial z}-\frac{\partial g(z_{II})}{\partial z}\right|\leqslant\left|\frac{\partial^2 g}{\partial z^2}\right||z_I-z_{II}|\leqslant\frac{40\,C}{K\delta^5}4C=\frac{160\,C^2}{K\delta^5}.$$

The last addend on the right-hand side of (26) is estimated by the use of inequalities (3), (5), (22), (18), and it does not exceed

$$\frac{32\,C}{K\delta^4}(4\,C+2\,C)<\frac{200\,C^2}{K\delta^4}.$$

So,

$$\left|\frac{\partial\hat{F}_1}{\partial\mu}\right|<C^2\left[\frac{4\cdot10^4}{K^2\delta^7}+2\pi\frac{160}{K\delta^5}+\frac{200}{K\delta^4}\right]<\frac{5\cdot10^4}{K^2\delta^7}C^2.$$

All these estimates are valid if the arguments z_I and z_{II} do not extend beyond the domain $|\text{Im}\,(z-2\pi\mu)|\leq R_0-2\delta$, where the estimates of g and its derivatives are in effect. Suffice it in this connection that $|\text{Im}\,z|\leq R_0-3\,\delta$. /62

Actually, then

$$|\widetilde{F}(z, \mu) + \widetilde{\Phi}(z, \mu, \Delta_0^*(\mu))| \leqslant 2C < \delta,$$

that is,

$$|\operatorname{Im}(z_1 - 2\pi\mu)| < R_0 - 2\delta.$$

Thus with $|\operatorname{Im} z| \leq R_0 - 3\delta$, $\mu \in N_K^r$, $|\Delta_1| < \delta^{27}$

$$\left|\frac{\partial \hat{F}_1}{\partial \mu}\right| < \frac{5 \cdot 10^4}{K^2 \delta^7} C^2. \tag{29}$$

6°. Estimate of $\frac{\partial}{\partial \mu}(\Delta - \Delta_0)$. We have:

$$\frac{\partial}{\partial \mu}(\Delta(\Delta_1, \mu) - \Delta_0^*(\mu)) = \frac{\partial \Delta(\Delta_1, \mu)}{\partial \mu} - \frac{\partial \Delta(0, \mu)}{\partial \mu};$$

according to the finite increment lemma,

$$\left|\frac{\partial}{\partial \mu}(\Delta - \Delta_0^*)\right| \leqslant \left|\frac{\partial^2 \Delta(\Delta_1, \mu)}{\partial \Delta_1 \partial \mu}\right| |\Delta - \Delta_0^*|.$$

We shall estimate $\left|\frac{\partial^2 \Delta(\Delta_1, \mu)}{\partial \Delta_1 \partial \mu}\right|$ by the use of the Cauchy integral as a derivative of $\frac{\partial \Delta}{\partial \mu}$. With $|\Delta_1| \leq \delta^{27}$, as it follows from (25), $\left|\frac{\partial \Delta}{\partial \mu}\right| < 4C$. Therefore, in circle $|\Delta_1| \leq \frac{\delta^{27}}{2}$ it is always

$$\left|\frac{\partial^2 \Delta}{\partial \Delta_1 \partial \mu}\right| < \frac{4C}{\frac{\delta^{27}}{2}} = 8.$$

In particular, $\left|\frac{\partial^2 \Delta}{\partial \Delta_1 \partial \mu}\right| < 8$, when $|\Delta_1| \leq \delta^{28}$. As

$$|\Delta - \Delta_0^*| \leqslant 2|\Delta_1|,$$

when $|\Delta_1| \leq \partial^{28}$, $\mu \in N_K^r$

$$\frac{\partial}{\partial \mu}(\Delta(\Delta_1, \mu) - \Delta_0^*(\mu))| < 16|\Delta_1|. \tag{30}$$

7°. Estimate of $\frac{\partial}{\partial\mu}|\tilde{\Phi}(\Delta(\Delta_1,\mu)) - \tilde{\Phi}(\Delta_0^*(\mu))|$. This derivative equals

$$\frac{\partial\tilde{\Phi}(\Delta)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\mu} + \frac{\partial\tilde{\Phi}}{\partial\Delta}\frac{\partial\Delta(\Delta_1)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}.$$

We shall estimate the first difference by the finite increment lemma: when $|\Delta| \leq \frac{\Delta_0}{2}$, $\mu \in N_K^r$, $|\text{Im } z| \leq R$

$$\left|\frac{\partial\tilde{\Phi}(\Delta)}{\partial\mu} - \frac{\partial\tilde{\Phi}(\Delta^*)}{\partial\mu}\right| \leqslant \left|\frac{\partial^2\tilde{\Phi}}{\partial\mu\partial\Delta}\right| |\Delta - \Delta_0^*| \leqslant 8\delta^2 |\Delta_1|$$

(here we estimated $\left|\frac{\partial^2\tilde{\Phi}}{\partial\mu\partial\Delta}\right|$ by the use of the Cauchy integral: $\left|\frac{\partial^2\tilde{\Phi}}{\partial\mu\partial\Delta}\right| < \frac{2\partial^2|\Delta_0|}{\frac{|\Delta_0|}{2}} \leq 4\delta^2$).

The second difference may be recorded as follows: /63

$$\frac{\partial\tilde{\Phi}(\Delta)}{\partial\Delta}\left(\frac{\partial\Delta(\Delta_1)}{\partial\mu} - \frac{\partial\Delta_0^*}{\partial\mu}\right) + \frac{\partial\Delta_0^*}{\partial\mu}\left(\frac{\partial\tilde{\Phi}(\Delta)}{\partial\Delta} - \frac{\partial\tilde{\Phi}(\Delta_0^*)}{\partial\Delta}\right), \tag{31}$$

where the first addend is estimated by the use of inequality (3) and does not exceed $16|\Delta_1|$, because $\left|\frac{\partial\tilde{\Phi}}{\partial\Delta}\right| < 1$ (see 2°), and the second addend by the use of the finite increment lemma, and it does not exceed

$$\left|\frac{\partial\Delta^*}{\partial\mu}\right| \left|\frac{\partial^2\tilde{\Phi}}{\partial\Delta^2}\right| |\Delta(\Delta_1) - \Delta_0^*| \leqslant 4C\frac{16}{\delta^{24}}2|\Delta_1|.$$

The only new feature here is the estimate of $\frac{\partial^2\tilde{\Phi}}{\partial\Delta^2}$. To find it, we used the expression for a second derivative obtainable from the Cauchy integral:

$$\left|\frac{\partial^2\tilde{\Phi}}{\partial\Delta^2}\right| \leqslant 2\frac{2\delta^2\Delta_0}{\left(\frac{\Delta_0}{2}\right)^2} = \frac{16}{\delta^{24}}$$

with $|\Delta| \leq \frac{\Delta_0}{2}$, the only requirement for which is, as we saw in 4°, the fulfillment of the inequality $|\Delta_1| \leq \delta^{27}$. Comparing all the three estimates, we find:

$$\left|\frac{\partial}{\partial\mu}[\widetilde{\Phi}(\Delta)-\widetilde{\Phi}(\Delta_0^*)]\right| < 8\delta^2|\Delta_1| + 16|\Delta_1| + 128\delta^2|\Delta_1|.$$

We finally get:

$$\left|\frac{\partial}{\partial\mu}[\widetilde{\Phi}(\Delta(\Delta_1,\mu))-\widetilde{\Phi}(\Delta_0^*(\mu))]\right| < 100|\Delta_1| \qquad (32)$$

with $|\Delta_1| \leq \delta^{28}$, $|\mathrm{Im}\, z| \leq R_0$, $\mu \in N_K^r$.

8°. Estimate of $\frac{\partial}{\partial\mu}\hat{\Phi}_1(z,\mu,\Delta(\Delta_1,\mu))$. It will be easier for us to begin by examining the function of z, μ and Δ_1, and not of z, μ, Δ. We have:

$$\frac{\partial\hat{\Phi}_1}{\partial\mu} = \left[\frac{\partial g(z_{III})}{\partial\mu} - \frac{\partial g(z_I)}{\partial\mu}\right] + \left[\frac{\partial g(z_{III})}{\partial z} - \frac{\partial g(z_I)}{\partial z}\right]\frac{\partial z_I}{\partial\mu} + \frac{\partial g(z_{III})}{\partial z}\left[\frac{\partial z_{III}}{\partial\mu} - \frac{\partial z_I}{\partial\mu}\right], \qquad (33)$$

where

$$z_I = z + 2\pi\mu + \overline{F}(z,\mu) + \widetilde{\Phi}(z,\mu,\Delta_0^*(\mu)), \qquad (27)$$

$$z_{III} = z + 2\pi\mu + \Delta_1 + \overline{F}(z,\mu) + \widetilde{\Phi}(z,\mu,\Delta(\Delta_1,\mu)) + \Delta_1. \qquad (34)$$

The first two brackets in the right part of (33) we shall estimate as in 5°:

$$\left|\frac{\partial g(z_{III})}{\partial\mu} - \frac{\partial g(z_I)}{\partial\mu}\right| \leqslant \left|\frac{\partial^2 g}{\partial\mu\,\partial z}\right| |z_{III} - z_I| \leqslant \frac{C\,10^4}{K^2\delta^7} 3|\Delta_1|,$$

as

$$z_{III} - z_I = \Delta_1 + \widetilde{\Phi}(z,\mu,\Delta) - \widetilde{\Phi}(z,\mu,\Delta^*(\mu))$$

and, in view of estimate (22) §5,

$$|z_{III} - z_I| \leqslant 3|\Delta_1|.$$

Analogically, /64

$$\left|\left(\frac{\partial g(z_{III})}{\partial z} - \frac{\partial g(z_I)}{\partial z}\right)\frac{\partial z_I}{\partial\mu}\right| \leqslant \left|\frac{\partial^2 g}{\partial z^2}\right| |z_{III} - z_I| \left|\frac{\partial z_I}{\partial\mu}\right| \leqslant$$
$$\leqslant \frac{40C}{K\delta^5} 3|\Delta_1| \left|2\pi + \frac{\partial\overline{F}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\mu} + \frac{\partial\widetilde{\Phi}}{\partial\Delta}\frac{\partial\Delta_0^*}{\partial\mu}\right| \leqslant \frac{40C}{K\delta^5} 3|\Delta_1|(2\pi + 6C) \leqslant \frac{1600C}{K\delta^5}|\Delta_1|,$$

where the multiplier $\left|\frac{\partial z_I}{\partial \mu}\right|$ is estimated by the use of condition 3) lemma 11 and estimate (22), bearing in mind that $C < 1$. We still have to estimate $\frac{\partial}{\partial \mu}(z_{III} - z_I)$. We have:

$$z_{III} - z_I = \Delta_1 + \tilde{\Phi}(z, \mu, \Delta(\Delta_1, \mu)) - \tilde{\Phi}(z, \mu, \Delta_0^\bullet(\mu)).$$

In view of estimate (32), we find:

$$\frac{\partial}{\partial \mu}(z_{III} - z_I) \leqslant 100 |\Delta_1|,$$

where $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$.

Thus,

$$\left|\frac{\partial g(z_{III})}{\partial z}\left(\frac{\partial z_{III}}{\partial \mu} - \frac{\partial z_I}{\partial \mu}\right)\right| \leqslant 100 |\Delta_1| \frac{32 C}{K\delta^4} \leqslant \frac{10^4 C}{K\delta^4} |\Delta_1|.$$

Comparing the estimates of all three addends in the right part of equality (33), we find:

$$\frac{\partial}{\partial \mu} \hat{\Phi}_1(z, \mu, \Delta(\Delta_1, \mu))\Big| \leqslant \frac{C\, 10^4}{K^2\delta^7} 3 |\Delta_1| + \frac{1600\, C}{K\delta^5} |\Delta_1| + \frac{C\, 10^4}{K\delta^4} |\Delta_1| \leqslant \frac{C\, 10^5}{K^2\delta^7} |\Delta_1|.$$

All these estimates are made on the assumption that $|\Delta_1| \leq \delta^{28}$, $\mu \in N_K^r$ and z_I, z_{III} do not extend beyond the band $|\mathrm{Im}\,(z - 2\pi\mu)| \leq R_0 - 2\delta$ where lemma 10 is in effect. All that is needed for this, for example, is that $|\mathrm{Im}\, z| \leq R_0 - 4\delta$, because then

$$|\Delta_1 + \tilde{F}(z, \varepsilon) + \tilde{\Phi}(z, \varepsilon, \Delta)| \leqslant \delta + 2C + 2C < 2\delta,$$
$$|\mathrm{Im}\,(z_{III} - 2\pi\mu)| \leqslant R_0 - 4\delta + 2\delta = R_0 - 2\delta.$$

9°. Estimate of $\frac{\partial z}{\partial \mu}$. The function of $g(z, \mu)$ is defined when

$$|\mathrm{Im}\,(z - 2\pi\mu)| \leqslant R_0 - 2\delta.$$

That means the following function is also defined in the same band:

$$\varphi(z, \mu) = z + g(z, \mu).$$

As in the mentioned band $|g(z, \mu)| \leq \delta$ [see (6), (17)], the shape of that band, at $z \to \phi$, will contain the band

$$|\operatorname{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta,$$

which, when $z \to \phi$, changes to a domain containing the band

$$|\operatorname{Im}(z - 2\pi\mu)| \leqslant R_0 - 4\delta.$$

It follows from 5.1 and 7.4 that

$$\frac{\partial z}{\partial \mu} = -\frac{\frac{\partial g}{\partial \mu}}{1 + \frac{\partial g}{\partial z}}.$$

According to inequality (18) and conditions 4) and 5) of lemma 11, /65
$\left|\frac{\partial g}{\partial z}\right| < \frac{1}{2}$, so that by using estimate (23), we get:

$$\left|\frac{\partial z}{\partial \mu}\right| \leqslant \frac{10^4 C}{K^2 \delta^6}$$

with $|\operatorname{Im}(z - 2\pi\mu) \leq R_0 - 2\delta$, $\mu \in N_K^r$ and, particularly, at

$$|\operatorname{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta.$$

10°. Estimate of $\frac{\partial}{\partial \mu} F_1(\phi, \mu)$, $\frac{\partial}{\partial \mu} \Phi_1(\phi, \mu, \Delta_1)$. According to 5.1,

$$F_1(\varphi, \mu) = \hat{F}_1(z(\varphi, \mu), \mu),$$
$$\Phi_1(\varphi, \mu, \Delta_1) = \hat{\Phi}_1(z(\varphi, \mu), \mu, \Delta(\Delta_1, \mu)).$$

The function of $z(\phi, \mu)$ is defined when $|\operatorname{Im}(\phi - 2\pi\mu)| \leq R_0 - 3\delta$, $\mu \in N_K^r$, and if

$$\operatorname{Im}(z - 2\pi\mu)| \leqslant R_0 - 4\delta,$$

then for this z there exists such ϕ that $z = z(\phi, \mu)$ and

$$|\operatorname{Im}(\varphi - 2\pi\mu)| \leqslant R_0 - 3\delta.$$

The functions of $\hat{F}_1(z)$, $\hat{\Phi}_1(z)$ are defined by $|\text{Im } z| \le R_0 - 4\delta$, and the functions of $F_1(\phi\ \mu)$, $\Phi_1(\phi, \mu, \Delta_1)$ are therefore defined by

$$|\operatorname{Im}\varphi| \leqslant R_0 - |\operatorname{Im} 2\pi\mu| - 5\delta$$

on the assumption that $|\text{Im } 2\pi\mu| \le R_0 - 5\delta$, that is, that $2\pi r \le R_0 - 5\delta$. In this domain

$$\frac{\partial F_1}{\partial\mu} = \frac{\partial\hat{F}_1}{\partial\mu} + \frac{\partial\hat{F}_1}{\partial z}\frac{\partial z}{\partial\mu}, \quad \frac{\partial\Phi_1}{\partial\mu} = \frac{\partial\hat{\Phi}_1}{\partial\mu} + \frac{\partial\hat{\Phi}_1}{\partial z}\frac{\partial z}{\partial\mu},$$

where, as in 8°, z, μ and Δ_1 are considered independent variables in the calculation of $\dfrac{\partial\hat{\Phi}_1}{\partial\mu}$.

We shall use the Cauchy integral to estimate $\dfrac{\partial\hat{F}_1}{\partial z}$ and $\dfrac{\partial\hat{\Phi}}{\partial z}$. Digressing by δ from the edge of the band where the estimates of F_1 and Φ_1 are known, we will find from the estimates in 3° and 5° §5:

$$\left|\frac{\partial\hat{F}_1}{\partial z}\right| \leqslant \frac{4C^2}{\delta^6}, \quad \left|\frac{\partial\hat{\Phi}_1}{\partial z}\right| \leqslant \frac{3C|\Delta_1|}{\delta^6}$$

at $|\text{Im } z| \le R_0 - 5\delta$; applying the estimates of 5°, 8° and 9°, we find from (35):

$$\left|\frac{\partial F_1}{\partial\mu}\right| \leqslant \frac{5\cdot10^4C^2}{K^2\delta^7} + \frac{10^4C}{K^2\delta^6}\frac{4C^2}{\delta^6},$$
$$\left|\frac{\partial\Phi_1}{\partial\mu}\right| \leqslant \frac{C\cdot10^5|\Delta_1|}{K^2\delta^7} + \frac{3C|\Delta_1|}{\delta^6}\frac{10^4C}{K^2\delta^6}.$$

Thus, at

$$|\Delta_1| \leqslant \delta^{28}, \quad \mu \in N_K^r, \quad 2\pi r \leqslant R_0 - 5\delta, \quad |\operatorname{Im}\varphi| \leqslant R_0 - |\operatorname{Im} 2\pi\mu| - 6\delta$$

we have:

$$|F_1(\varphi, \mu)| \leqslant \frac{C^2}{\delta^6}, \qquad |\Phi_1(\varphi, \mu, \Delta_1)| < \frac{C|\Delta_1|}{\delta^6},$$
$$\left|\frac{\partial F_1}{\partial\mu}\right| \leqslant \frac{C^2}{\delta^{13}}, \qquad \left|\frac{\partial\Phi_1}{\partial\mu}\right| \leqslant \frac{C|\Delta_1|}{\delta^{13}}$$

because /66

$$\frac{5\cdot10^4}{K^2}\delta < 1. \tag{6}$$

In the same way, all the other estimates of 1° - 9° can be expressed as in (7) - (16), in view of conditions 4) and 5) of lemma 11.

Lemma 11 has been proved.

8.3. Proof of theorem 3. Theorem 3 is deduced from lemma 11, just as theorem 2 was deduced from the basic lemma in §6.

We shall select $\delta_1 > 0$ so that

1) $\sum_{n=1}^{\infty} \delta_n < \frac{R}{8}$, where $\delta_n = \delta_{n-1}^{1\frac{1}{2}}$ $(n = 2, 3, \ldots)$,

2) $\delta_1 < \frac{K^2}{5 \cdot 10^4}$.

Let $R = R_0$ and K the same, which under the terms of theorem 2, $\mu \in N_K^{\frac{R}{16\pi(a+1)}}$, $\Delta_0 = \delta_1^{26}$, $L\epsilon_0 < C_1$ where

$$C_1 = \delta_1^{27}, \tag{35}$$

and C_1, δ_1 are, respectively, C and δ of lemma 11. We then get from inequalities (7) - (16):

$$|F_1| < \frac{\delta_1^{54}}{\delta_1^{13}} < \delta_1^{40,5} = (\delta_1^{1\frac{1}{2}})^{27} = \delta_2^{27},$$

$$\left|\frac{\partial F_1}{\partial \mu}\right| < \delta_2^{27},$$

$$|\Phi_1| \leqslant \frac{\delta_1^{27}}{\delta_1^{13}} |\Delta_1| < \delta_1^{3} |\Delta_1| = \delta_2^{2} |\Delta_1|,$$

$$\left|\frac{\partial \Phi_1}{\partial \mu}\right| < \delta_2^{2} |\Delta_1|$$

with

$$\Delta_1| < \delta_2^{26} = \delta_1^{39} < \delta_1^{28}, \quad |\operatorname{Im} \varphi_1| \leqslant R_0 - 7\delta_1 - |\operatorname{Im} 2\pi\mu| = R_1, \;\; \mu \in N_K^{\frac{R}{16\pi(n+1)}}.$$

Thus, we find ourselves again under the terms of lemma 11 but with a radius R_1 reduced by $7\delta_1 + 8\frac{R}{(n+1)}$. As

$$\sum_{n=1}^{\infty} \delta_n < \frac{R}{8},$$

we will be able to make an n-number of successive approximations, and the last one will be effective when

$$|\mathrm{Im}\,\varphi_n| \leqslant \frac{R}{8(n+1)}, \quad \mu \in N_K^{\frac{R}{16\pi(n+1)}}, \quad |\Delta_n| < \delta_{n+1}^{26}.$$

Omitting the usual proof (see §6) of the convergence of approximations at real μ, we shall estimate $\left|\frac{\partial \Delta(n)}{\partial \mu}\right|$.

It follows from 8.1 that /67

$$\frac{\partial \Delta_k^{(n)}}{\partial \mu} = \frac{\partial \Delta_k}{\partial \mu} + \frac{\partial \Delta_k}{\partial \Delta_{k+1}} \frac{\partial \Delta_{k+1}^{(n)}}{\partial \mu}.$$

Assuming $C_k = \delta_k^{27}$, we find on the basis of lemma 11:

$$\left|\frac{\partial \Delta_k^{(n)}}{\partial \mu}\right| \leqslant 4C_{k+1} + 2\left|\frac{\partial \Delta_{k+1}^{(n)}}{\partial \mu}\right|.$$

If

$$\left|\frac{\partial \Delta_{k+1}^{(n)}}{\partial \mu}\right| < C_{k+1},$$

then

$$\left|\frac{\partial \Delta_k^{(n)}}{\partial \mu}\right| < 6C_{k+1} < C_k.$$

As

$$\left|\frac{\partial \Delta_n^{(n)}}{\partial \mu}\right| = 0,$$

then

$$\left|\frac{\partial \Delta_0^{(n)}}{\partial \mu}\right| < 6C_1.$$

Theorem 3 has been proved.

Remark. The monogenetic aspects of functions g_n, F_n, Φ_n, ϕ_n could be proved, and analogical estimates obtained, in the same way.

PART II.

Concerning the Space Representation of a Circle

The problem of studying the rotation number-equation factor dependence was raised by Poincare (1). The treatment of the rotation number as a function in mapping space helps clarify the problem of typical and exceptional cases.

We shall disignate the angular coordinates of the points on a circle by lower-case Greek letters; ϕ and $\phi + 2\pi$ represent the same point of a circle. The transformations will be designated by capital letters:

$$T : \phi \to T\phi.$$

We shall discuss only continuous, mutually single-valued direct (orientation retaining) transformations. A rotation to angle $\theta : \phi \to \phi + \theta$ could serve as an example. For every transformation there is a "shift," a function on the circle showing how far each point is shifted. We shall designate the shift by the same letter as the transformation, only with a lower-case letter:

$$T : \phi \to T_{\phi} = \phi + t(\phi).$$

Here $t(\phi)$ represents the shift. If T is a rotation to zero angle, then $t(\phi) \equiv 0$. Generally speaking, the shift, just like ϕ, is defined only correct to the multiple 2π; however, having defined $t(\phi)$ at one point, we can extend it unilaterally along the continuity.

If T is a smooth transformation, then $t(\phi)$ is a smooth periodic function: /68

$$t(\varphi + 2\pi) = t(\varphi).$$

We shall designate as

$$T^n\varphi = \varphi + t^{(n)}(\varphi)$$

the n-th degree of transformation T. By this designation it is assumed that branch $t^{(n)}(\phi)$ was selected to correspond to branch $t(\phi)$;

$$t^{(n)}(\varphi) = t^{(n-1)}(\varphi) + t(T^{n-1}(\varphi)) \quad (n = 2, 3, \ldots).$$

Under these terms $t^{(n)}\,\phi$ is called a displacement of n steps.

§9. The Function of μ (T) and Its Level Sets

Let us examine the spaces

$$C \supset C^1 \supset C^2 \supset \ldots \supset C^n \supset \ldots \supset C^\infty \supset A$$

of mutually single-valued direct representations of a circle, that is, continuous and continuously and infinitely differentiable and analytic representations in the neighborhood of a real axis with a topology usual in these spaces. Each successive topology is stronger than the preceding one, and each of the spaces is absolutely dense within the preceding one.*

Poincare (1) defined the rotation number $2\pi\mu$ for every transformation $T \in C$; thus the function μ (T) is given for space C. The following theorem was suggested by Poincare without proof.

THEOREM 4. Function μ(T) is continuous at every point C.

Proof. We will show that μ (T) is continuous at point T_0.

Let $\epsilon > 0$. We will select integer $n > \frac{2}{\epsilon}$ so that

$$\frac{m}{n} < \mu(T_0) < \frac{m+1}{n}.$$

Then in the following transformation

$$T_0^n : \varphi \to \varphi + t_0^{(n)}(\varphi)$$

each point will be shifted by more than 2π m. Actually, if some points were displaced by less and others by more than 2πm, there would also be a point displaced by exactly 2πm, that is, stationary for T_0^n; it is obvious, therefore, that despite the selection of n, we would have

$$\mu = \frac{m}{n}.$$

If all the points were shifted by less than 2πm, we would get $\mu < \frac{m}{n}$, which again contradicts the selection of n.

*If T is included in one of the spaces C^1, C^2, ..., A, regardless of which particular space, we shall T a smooth transformation.

It can similarly be proved that each point is displaced by n steps less than $2\pi(m+1)$. So,

$$2\pi m < t_0^{(n)}(\varphi) < 2\pi(m+1).$$

In view of the continuity $t_0^{(n)}(\phi)$, /69

$$2\pi m + \eta < t_0^{(n)}(\varphi) < 2\pi(m+1) - \eta$$

at some $\eta > 0$, and in view of the continuous $T^n - T$ dependence, there will be $\delta > 0$, such that

$$|t^{(n)}(\varphi) - t_0^{(n)}(\varphi)| < \eta,$$

as soon as transformation T differs from T_0 by less than δ:

$$|t(\varphi) - t_0(\varphi)| < \delta.$$

For such T

$$2\pi m < t^{(n)}(\varphi) < 2\pi(m+1)$$

and, therefore,

$$\frac{m}{n} < \mu(T) < \frac{m+1}{n}.$$

So, $|\mu(T) - \mu(T_0)| < \epsilon$ at $|t(\phi) - t_0(\phi)| < \delta$. The theorem has been proved.

Remark. Even in the best cases, function $\mu(T)$ is always continuous. Let us examine a family of transformations, for example,

$$T_h: \varphi \to \varphi + h + 0{,}1\sin^2\varphi,$$

where h stands for the parameter. As has been proved, $\mu(T_h)$ is a continuous function of h. The function $\mu(T_h)$ increases with increasing h, but is retarded by each rational value of μ: corresponding to it is an entire segment $(h_1 h_2)$ of h values. But with $h > h_2$, the $\mu(T_h)$ function increases very rapidly: E. G. Bellaga showed that in the neighborhood of zero, for example, $\mu(T_h)$ increases at least as $\frac{C\sqrt{h}}{-\log h}$.

The sets of level μ (T) are multiple transformations with the same rotation number $2\pi\mu$. Such transformations include the rotation to angle $2\pi\mu$, the transformations converted to a rotation to angle $2\pi\mu$ by the proper substitution of a variable and, possibly, other transformations.

The structure of the sets of level $\mu(T) = \mu$ depends a great deal on whether μ is rational or irrational.

§10. The Case of a Rational μ

10.1. If $\mu(T) = \frac{m}{n}$, then, as Poincare showed, T^n has the stationary points $t^{(n)}(a) = 2\pi m$. Their set is invariant in relation to T and closed, as a set of the continuous function level $t^{(n)}(a)$. The points a, T_a, ..., T^{n-1} a are called a cycle. To investigate a cycle, it would be useful to examine the transformation T^n graph and function $t^{(n)}(\phi)$ graph (see Fig. 7; that figure shows an outline of the $T(\phi) = \phi + \frac{1}{2}\cos\phi$ graph, and the forms of 0 in connection with some iterative T). This /70 cycle is called isolated if in the neighborhood of its points there are no points of other cycles. An isolated cycle is stable if its point (which also means all its points) has an indefinite number of small neighborhoods which are transferred into themselves (Russian term: perekhodyashchiye vnutr' sebya) during the transformation T^n. It is easy to see that when $n \to +\infty$, the points of such a neighborhood gravitate towards the points of a cycle, which explains the term. A stable transformation cycle T^{-1} is called the unstable cycle T. An isolated cycle is semistable forward (backward) if all the points in some neighborhood of the cycle point (except the point itself) are shifted by transformation T^n forward (backward), that is, if in this neighborhood

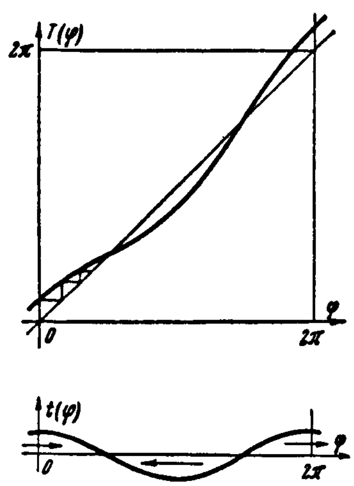

Fig. 7

$$t^{(n)}(\varphi) - 2\pi m > 0 \quad (< 0).$$

The transformation $T \in C^1$ is normal, if at the points of its cycles

$$\frac{dt^{(n)}(\varphi)}{d\varphi} \neq 0.$$

Obviously, a normal transformation has a finite number of cycles, and all its cycles are stable or unstable. It is the roots $t^{(n)}(\phi) - 2\pi m$, where $\frac{dt^{(n)}}{d\phi} < 0$ that are points of stable cycles, and those with $\frac{dt^{(n)}}{d\phi} > 0$ are points of unstable cycles. Hence, all the points of stable and unstable cycles of a normal transformation are intermittent.

10.2. THEOREM 5. Normal transformations form a set which is open in C^1 and absolutely dense in A.

Proof. 1. The points of a cycle are those where $t^{(n)}(\phi) = 2\pi m$. In them $\frac{dt^{(n)}(\phi)}{d\phi} \neq 0$. Therefore, in the case of a small change $t^{(n)}(\phi)$ together with the first derivative, the function $t^{(n)}(\phi) - 2\pi m$ does not acquire any new roots, and the old ones do not disappear but are continually displaced, and the derivative in the root retains its sign. That means that transformation T with such a changed function $t^{(n)}(\phi)$ will be normal. In view of the continuous $T^{(n)}(\phi) - T$ dependence, the first assertion of the theorem has been proved.

2. We will show that there is an analytic transformation with a cycle in any proximity to any transformation. Obviously, such proof will be sufficient for an analytic transformation and analytic proximity. Let T be an analytic transformation with an irrational rotation number, and let $\epsilon > 0$. Among the points $\phi_n = T^n\phi_0$ there is one removed back from ϕ_0 by less than ϵ, for example:

$$2\pi m - \varepsilon < t^{(n)}(\varphi_0) < 2\pi m$$

(the Denjoy theorem). Let us examine a family of analytic transformations T_λ ($\lambda \geq 0$, $T_0 = T$): /71

$$T_\lambda : \varphi \to \varphi + t(\varphi) + \lambda.$$

It is easy to see that, with $\lambda = \epsilon$, T_λ^n shifts ϕ_0 forward:

$$t_\lambda^{(n)}(\varphi_0) \geqslant 2\pi m.$$

Hence, in view of the continuity of $t_\lambda^{(n)}(\phi_0)$ for λ, it follows that at some $\lambda_0 \leq \epsilon$, T_{λ_0} has a cycle ϕ_0, $T_{\lambda_0}\phi_0$, ... :

$$t_{\lambda_0}^{(n)}(\varphi_0) = 2\pi m.$$

3. An analytic transformation with a cycle can be converted to a normal one by an infinitely small change. Indeed, let T be an analytic transformation with no stable cycles (which also means no unstable cycles). We shall select cycle $\phi_0, \phi_1, \ldots, \phi_{n-1}$ and introduce the analytic function of $\Delta(\phi)$ which becomes zero at these points, having a negative derivative in them. The transformation

$$T_\theta : t_\theta(\varphi) = t(\varphi) + \theta\Delta(\varphi)$$

with a small θ proximate to T, has at least one stable cycle $\phi_0, \phi_1 \ldots,$ ϕ_{n-1}. All we have to do, therefore, is to examine a case in which the initial transformation T has a stable cycle. We shall construct an analytic function of $\delta(\phi)$, with respect to T, which

1) is equal to zero and has a negative (positive) derivative at the points of the stable (unstable) cycles of T;

2) is positive (negative) at the points of the T cycles which are semistable forward (backward).

The existence of such a function is obvious, as the number of cycles of T is finite, because the analytic function of $t^{(n)}(\phi) - 2\pi m$ has an isolated root and is not therefore an identical zero.

Let us look at the transformation T_θ: $\phi \to \phi + t(\phi) + \theta\delta(\phi)$. With a small θ, this transformation is normal; the formal proof that the stable cycles of T with small θ are only somewhat displaced, that the roots of $t^{(n)}(\phi) - 2\pi m$ become simple numbers, and that the semistable cycles disappear is left up to the reader. When θ is small enough, the transformation T_θ is the unknown quantity.

Theorem 5 has been proved.

10.3. The construction of a normal transformation can be easily observed on the graph of function $t^{(n)}(\phi) - 2\pi m$. Its roots, the points of the transformation cycles, divide the circle into arcs. Each arc $\alpha\beta$ is bounded on one side by point α of a stable cycle and, on the other, by point β of an unstable cycle. With $n \to +\infty$, the points of the arc are wound onto the stable cycle, and, with $n \to -\infty$, onto the unstable cycle, that is,

$$\lim_{k\to\infty} T^{kn}(\gamma) = \alpha \pmod{2\pi}, \quad \lim_{k\to-\infty} T^{kn}(\gamma) = \beta \pmod{2\pi},$$

where $\gamma \in (\alpha, \beta)$. Such assertions are well known in the qualitative theory of differential equation, and we will omit their proof.

Thus a topologically normal transformation is characterized by three integers: m, n, k, where $\frac{m}{n}$ is the rotation number and k the number of stable (which means also unstable) cycles. Two transformations with the same m, n, k are arranged in the same way in a sense that one of them can be converted to the other by a continuous change of a variable on the circle (that is, $T_2 = \Phi T_1 \Phi^{-1}$, where $\Phi \in C$). The invariant of smooth change of a variable is also a derivative of $\frac{dt^{(n)}(\phi)}{d\phi}$ in the points of the cycle characterizing the speed of (winding) onto the cycle. There are probably no other invariants in existence, but I was unable to prove that. /72

THEOREM 6. The set $E_{\frac{m}{n}}$ of level $\mu = \frac{m}{n}$ in any of the spaces C^1, ..., A is compendent and consists of

1) a normal transformation nucleus $\bigcup_{k=1}^{\infty} E^k_{\frac{m}{n}}$ which is dense in $E_{\frac{m}{n}}$ and open in C^p (A). The nucleus consists of compendent transformation components $E^k_{\frac{m}{n}}$ with k stable and k unstable cycles. Two transformations of the same $E^k_{\frac{m}{n}}$ component can be converted to one another by a continuous change of a variable;

2) the boundaries of $E_{\frac{m}{n}}$ and $E^k_{\frac{m}{n}}$. The $E_{\frac{m}{n}}$ boundary consists of T transformations where $t^{(n)}(\phi) - 2\pi m$ does not change the sign. Its parts F_+ $(t^{(n)}(\phi) - 2\pi m \geq 0)$ and F_- $(t^{(n)}(\phi) - 2\pi m \leq 0)$ contain semi-stable (forward and backward) transformations, are compendent and intersect along the compendent set of F_0. The transformations from F_0 are converted by a smooth change of variable to a rotation. F_0 is included in the boundary of every $E^k_{\frac{m}{n}}$ component.

Proof. 1. The sets $E_{\frac{m}{n}}$, F_+, F_- are compendent. To prove it, we will connect, within the domain of the given set, any transformation $T \in E_{\frac{m}{n}}$ (F_+, F_-) with rotation T_2 to angle $2\pi\frac{m}{n}$ by the arc T_θ $(0 \leq \theta \leq 2$, $T_0 = T)$. Let $\phi_0, \ldots, \phi_{n-1}$ be the cycle of T. By a smooth change of a variable

$$\varphi \rightarrow \Psi\varphi = \varphi + \psi(\varphi)$$

we will convert the points $\phi_0, \ldots, \phi_{n-1}$ to $2\pi\frac{m}{n}\underline{1}$ $(0 \leq \underline{1} \leq n-1)$. Let us assume that

$$\Psi_\theta\varphi = \varphi + \theta\psi(\varphi)$$

and examine

$$T_\theta\varphi = \Psi_\theta T \Psi_\theta^{-1}\varphi = \varphi + t_\theta(\varphi) \quad (0 \leqslant \theta \leqslant 1).$$

This transformation is a transformation of T recorded in the variable Ψ_θ, and belongs to $E_{\frac{m}{n}}$ (F_+, F_-).

Let us examine the line segment connecting T_1 and T_2: /73

$$T_\theta\varphi = \varphi + (\theta - 1)\, 2\pi\frac{m}{n} + (2 - \theta)\, t_1(\varphi) \quad (1 \leqslant \theta \leqslant 2).$$

The points $2\pi\frac{m}{n}\underline{1}$ $(0 \leq \underline{1} \leq n-1)$ form a cycle T_0 with all $1 \leq \theta \leq 2$, and the T_0 line therefore lies entirely within $E_{\frac{m}{n}}$ $(F_+, F_-$, respectively). The connectivity has been proved.

2. The set $E^k_{\frac{m}{n}}$ of normal transformations with known m, n, k is compendent in any of the $C^1, \ldots$, A spaces. To prove it, we will connect the transformations T_0, T_2 by an arc T_θ $(0 \leq \theta \leq 2)$ in a selected space. We shall follow up by a smooth change of a variable

$$\Psi\varphi = \varphi + \psi(\varphi),$$

that will transfer the points of cycles T_0 to the corresponding points of the T_2 cycles (which can easily be done as the number of these points is the same, and they follow in the same order). The transformation $T_1 = \Psi T_0 \Psi^{-1}$ affects the points of cycles T_2 as a transformation of T_2; it can readily be seen that it has no other cycles. Assuming that

$$\Psi_\theta(\varphi) = \varphi + \theta\psi(\varphi)$$

and

$$T_\theta = \Psi_\theta T_0 \Psi_\theta^{-1} \quad (0 \leqslant \theta \leqslant 1),$$

we will connect T_0 with T_1 by a curve lying within $E^k_{\frac{m}{n}}$.

Let us look at the following transformations:

$$T_1(\varphi) = \varphi + t_1(\varphi), \quad T_2(\varphi) = \varphi + t_2(\varphi).$$

The functions t_1 (ϕ) and t_2 (ϕ) coincide at the cycle points, and all the transformations therefore

$$T_\theta(\varphi) = \varphi + (2-\theta)\, t_1(\varphi) + (\theta - 1)\, t_2(\varphi) \quad (1 \leqslant \theta \leqslant 2)$$

have the same cycles. Consequently, the line T_θ $(0 \leq \theta \leq 2)$, connecting T_0 with T_2, lies entirely within $K^k_{\frac{m}{n}}$.

3. The proof that the set $E^k_{\frac{m}{n}}$ is open and that the set $\bigcup_k E^k_{\frac{m}{n}}$ of normal transformations with an $\frac{m}{n}$ rotation number is absolutely dense in $E_{\frac{m}{n}}$ is similar to the proof of theorem 5 (points 1 and 3).

4. If T_1, $T_2 \in E^k_{\frac{m}{n}}$, it is possible to make a continuous change of variable $\Psi = \phi + \Psi\,(\phi)$ such that T_1 will change to T_2 : $T_2 = \Psi T_1 \Psi^{-1}$. Indeed, we shall designate the points of the stable cycles T_1 as $a^{\underline{l}}_i$ $(1 \leq \underline{l} \leq k,\ 1 \leq i \leq n,\ T_1 a_i = a_{i+1},\ a_{n+1} = a_1)$, and the points of the unstable cycles T_1 as $b^{\underline{l}}_i$ (we will use $\underline{l}$ to designate the number of the cycle in the sequence on the circumference). In this case there are no cycle points on the $a^1_1 b^1_1$ arc (and that means that the same applies to every $a^{\underline{l}}_i b^{\underline{l}}_i$ and $b^{\underline{l}}_i a^{\underline{l}+1*}_i$ arc). /74

Further, let $c^{\underline{l}}_i$ and $d^{\underline{l}}_i$ be similarly numbered points of stable and unstable cycles T_2. The substitution of variable Ψ changes the points $a^{\underline{l}}_i$, $b^{\underline{l}}_i$ to $c^{\underline{l}}_i$, $d^{\underline{l}}_i$, and we still have to complete the definition of Ψ on the $a^{\underline{l}}_i b^{\underline{l}}_i$ and $b^{\underline{l}}_i a^{\underline{l}}_i$ arcs. We will select points x and y within the arcs $a^1_1 b^1_1$ and $c^1_1 d^1_1$. Points $T^n_1 x$ and $T^n_2 y$ lie on the same arcs closer to a^1_1 and c^1_1, respectively. We will use Ψ to map an arc $(x, T^n_1 x)$ onto an arc $(y, T^n_2 y)$ by the homomorphic and direct methods: $x \to y$, $T^n_1 x \to T^n_2 y$.

*With $\underline{l} = k$, $\underline{l} + 1$ implies 1.

Obviously, in the transformations of T_1^p the image of arc $[x, T_1^n x]$ (and arc $[y, T_1^n y]$ in the transformation of T_2^p) will cover the entire arcs $a_i^1 b_i^1$ $(1 \leq i \leq n)$ (as well as all the $c_i^1 d_i^1$ arcs). We thus define $\Psi(\phi)$ on arc $T_1^p x$, $T_1^{p+n} x$ as $T_2^p \Psi T_1^{-p}$. A similar construction is possible on arc $a_i^1 b_i^1$ and $b_i^1 a_i^{1+1}$. The proof that the found substitution of a variable was the unknown quantity is a simple one, and we will therefore omit it.

5. The construction of boundaries. If $T^{(n)}(\phi) - 2\pi m$ changes its sign, then T is an internal point of $E_{\frac{m}{n}}$, because with a slight change of T $t^{(n)}(\phi)$, it will continue to change the sign as before, and T will retain the cycle. The boundary of $E_{\frac{m}{n}}$ is therefore included in the sum of F_+ ($T \in F_+$, if $t^{(n)}(\phi) - 2\pi m \geq 0$) and F_-. To convert the transformation of $T \in F_0 = F_+ \cap F_-$ to a rotation, we must change the points of one cycle to $2\pi \frac{m}{n} \underline{1}$ by a smooth change of a variable, and then redefine the parameter on all the arcs $\left[2\pi \frac{ml}{n}, 2\pi \frac{ml+1}{n}\right]$ with the exception of one ($\underline{1} = 0$), according to formula

$$\Psi(\varphi) = 2\pi \frac{ml}{n} + T^{-l}(\varphi).$$

A small change of the rotation to angle $2\pi \frac{m}{n}$ may change it to a transformation from any $E^k_{\frac{m}{n}}$, just as was done in the proof of theorem 5 (point 3). It follows from the previous argument that the same holds true for all transformations from F_0, which proves the last assertion of theorem 6.

10.4. It follows from theorem 6 (point 4 of the proof) that normal transformations are crude in the sense of Andronov-Pontryagin [10]. Since, according to theorem 5, the set of all normal transformations is absolutely dense, no abnormal transformation can be approximate.

From a topological point of view, normal transformations fill an overwhelming part of the transformation space - an absolutely dense open set. It will be shown in the next paragraph that an ergodic case is also typical from the point of view of measure. /75

§ 11. The Case of an Irrational μ.

11.1. Let us examine a set E_μ of an irrational μ level. According to the Denjoy theorem, every transformation of $T \in E_\mu$ in the C^2, ..., A spaces can be changed to a rotation to angle $2\pi\mu$ by a continuous change of a variable. But we are interested in a transformation changing to a rotation by a smooth change of a variable. We will designate a set of such transformations by E_μ^{Cp} (and by E_μ^A; general designation E'_μ).

THEOREM 7. 1°. The set E_μ^A is absolutely dense in E_μ, according to topology C. All sets E'_μ are compendent.

2°. If μ is such that $|\mu - \frac{m}{n}| > \frac{K}{|n|^3}$ with any integers m and $n \neq 0$, the E_μ^A set is open in E_μ, according to topology A.

Proof. 1°. Let T_0 denote a rotation to angle $2\pi\mu$, and let $T_1 \in E'_\mu$. Then there exists a smooth substitution of a variable

$$\Psi(\varphi) = \varphi + \psi(\varphi)$$

such that $T_1 = \Psi T_0 \Psi^{-1}$. The substitution of

$$\Psi_\theta(\varphi) = \varphi + \theta\psi(\varphi) \quad (0 \leqslant \theta \leqslant 1)$$

changes T_0 to $T_0 = \Psi_0 T_0 \Psi_0^{-1}$; thus the line T_θ, connecting T_0 with T_1, lies entirely in E'_μ. The connectivity E'_μ has been proved.

We will construct in E_μ^A a transformation T^* in a prescribed neighborhood $T \in E_\mu$. According to the Denjoy theorem, there is a continuous change of variable $\Psi(\phi)$, such that $T = \Psi T_0 \Psi^{-1}$. We will construct an analytic change of $\Psi^*(\phi)$ by variable (ϕ) so that Ψ and Ψ^*, Ψ^{-1} and Ψ^{*-1} are little different in metric C. Then $T^* = \Psi^* T_0 \Psi^{*-1}$ will approximate T in metric C and belong to E_μ^A. The assertion 1° has been fully proved.

2°. The fact that the set E_μ^A is open in $E_\mu \cap A_1$ follows from theorem 2. Obviously, all that has to be shown is that some neighborhood of rotation T_0 in $E_\mu \cap A$ is included in E_μ^A. The transformation $T \in E_\mu \cap A$ may be written as

$$\varphi \to \varphi + 2\pi\mu + F(\varphi),$$

and the neighborhood $U_{R,\ C}$ of transformation T_0 is defined by the inequality $|F(\phi)| < C$ with $|\mathrm{Im}\ \phi| < R$. But in view of theorem 2 (see point 4.3), the given R is accompanied by a C such that all transformations $T \in U_{R,\ C} \cap E_\mu$ are analytically reduced to a rotation. Theorem 7 has been proved.

11.2. Approaching the problem of typicalness from the point of view of measure (see [8]), we discover the lack of a sensible measure in functional spaces, and are therefore compelled to confine ourselves to finite-dimensional spaces.

Let us examine a two-dimensional space of analytic transformations /76

$$A_{a,b}: z \to z + a + F(z, b),$$

where, given $|\mathrm{Im}\ z| < R$, $|b| < b_0$, $F(z, b)$ is an analytic function satisfying inequality $|F(z, b)| < L|b|$.

THEOREM 8.

$$\lim_{\theta \to 0} \frac{\mathrm{mes}\ E_\theta}{2\pi\theta} = 1, \tag{1}$$

where E_θ is a set of plane points (ab), $a \in [0, 2\pi]$, $b \in [0, \theta]$, such that transformation A_{ab} changes to a rotation by an analytic substitution of coordinate z.

Proof. 1. Let us take a look at set M_k, a compact set of points $0 < \mu < 1$, that satisfies inequality

$$\left|\mu - \frac{m}{n}\right| \geqslant \frac{K}{n^3}$$

with all $m, n > 0$. According to theorem 2, for any $\mu \in M_K$ there exists $C = C(K, R) > 0$ and an analytic function $\Delta(b, \mu)$ for b such that the transformations $A_{2\pi\mu + \Delta(b, \mu),\ b}$ at $\mu \in M_K$, $|b| < C$ can be changed to a rotation by an analytic change of the parameter: $(2\pi\mu + \Delta(b, \mu), b) \in E_\theta$. We will use $M_K(b)$ to designate the set of points $\mu + \frac{\Delta(b, \mu)}{2\pi}$, $\mu \in M_K$, with a fixed b. Then the transformation $D_b : \mu \to \mu + \frac{\Delta(b, \mu)}{2\pi}$ will change M_K to set $M_K(b)$.

We will assume that $\epsilon > 0$, and select $K > 0$ so that mes $M_{2K} > 1 - \frac{\epsilon}{3}$ (this is possible, according to lemma 1 § 2). We will show that when the b value is small enough, the following inequality is valid

$$\text{mes } M_{\frac{K}{2}}(b) > 1 - \varepsilon,$$

and its immediate result will be theorem 8, because it is obvious that

$$2\pi\theta \geqslant \text{mes } E_\theta \geqslant 2\pi \int_0^\theta \text{mes } M_{\frac{K}{2}}(b)\, db.$$

2. Shown in § 7 is a perfect set $N_K^0 = N_K$, $M_{2K} \subseteq N_K \subseteq M_{\frac{K}{2}}$.

Obviously, all that has to be shown is that when b is sufficiently small

$$\text{mes } N_K(b) > 1 - \varepsilon. \tag{2}$$

(Inasmuch as $K > 0$ is fixed, we will now omit index K: $N_K = N$.)

According to theorem 3, the representation $D_b : N \to N\,(b)$ is the limit of a uniformly converging sequence of instantaneous representations

$$D_b^n : \mu \to \mu + \frac{1}{2\pi} \Delta^n (b, \mu).$$

We will show that for any $\epsilon > 0$ there will be found a $b\,(\epsilon)$ such that, with $b < b\,(\epsilon)$ and any n

$$\text{mes } D_b^n(N) > 1 - \varepsilon. \tag{3}$$

In view of theorem 3, there will be found $b\,(\epsilon)$ such that with all n, $b < b\,(\epsilon)$, $\mu \in N$ the following inequality will be valid /77

$$\left| \frac{\partial \Delta^n}{\partial \mu} \right| < \frac{\varepsilon}{3},$$

that is, in the representation of D_b^n, N can be mapped almost without expansion.

We will prove that this $b\,(\epsilon)$ was the unknown quantity (the index n will be omitted everywhere, as we are now dealing with a fixed n). Let

$b < b\ (\epsilon)$. By the definition of monogeneity, for every $\frac{\epsilon}{3}$ there will be $\delta > 0$, such that

$$\left|\frac{\Delta(\mu_1) - \Delta(\mu_2)}{\mu_1 - \mu_2} - \frac{\partial\Delta(\mu_3)}{\partial\mu}\right| < \frac{\varepsilon}{3},$$

if $|\mu_1 - \mu_3| < \delta, |\mu_2 - \mu_3| < \delta, \mu_1, \mu_2, \mu_3 \in N$. Then under the same conditions

$$\left|\frac{\Delta(\mu_1) - \Delta(\mu_2)}{\mu_1 - \mu_2}\right| < \frac{2\varepsilon}{3}, \quad (4)$$

according to selection $b\ (\epsilon)$.

3. We will divide N into nonintersecting (but measurable) parts N^i, $\bigcup_{i=1}^{L} N^i = N$, each with a diameter less than δ, and let $N^i\ (b)$ be their images in the transformation D_b^n. Since in this transformation the distance between two points N^i may be reduced, according to (4), not more than $1 - \frac{2\epsilon}{3}$ times, then

$$\operatorname{mes} N^i(b) > \left(1 - \frac{2\varepsilon}{3}\right) \operatorname{mes} N^i,$$

hence:

$$\sum_{i=1}^{L} \operatorname{mes} N^i(b) > \left(1 - \frac{2\varepsilon}{3}\right) \sum_{i=1}^{L} \operatorname{mes} N^i.$$

Thus,

$$\operatorname{mes} N(b) > \left(1 - \frac{2\varepsilon}{3}\right) \operatorname{mes} \Lambda,$$

and, as

$$\operatorname{mes} N > 1 - \frac{\varepsilon}{3},$$

we get:

$$\operatorname{mes} N(b) > \left(1 - \frac{2\varepsilon}{3}\right)\left(1 - \frac{\varepsilon}{3}\right) > 1 - \varepsilon,$$

and inequality (3) has been proved. Its corollary is inequality (2), because the following is valid.

LEMMA. Let $E \subseteq [0, 1]$ be a perfect set, f_n the sequence of its continuous representations on $F_n \subseteq [0, 1]$ uniformly converging to representation $f : E \to F$, and let $0 \leq \Delta < 1$. If mes $F_n > 1 - \Delta$, with all n, then mes $F \geq 1 - \Delta$.

Proof. Let $\epsilon > 0$. Let us examine set D_ϵ of the contiguous intervals F exceeding ϵ. There will be a finite number of them and, with n large enough, these intervals will differ very little from the corresponding contiguous intervals F_n. The total length of the latter, with any n, is less than Δ, as $F_n > 1 - \Delta$. The total length of D_ϵ therefore does not exceed Δ. In view of the arbitrary nature of $\epsilon < 0$, the entire addition to F will not exceed Δ either, which is what had to be proved. /78

Assuming that $E = N$, $f_n = D_b^n$, $F_n = D_b^n(N)$, $\Delta = \epsilon$, we will get inequality (2) from (3). Theorem 8 has been proved.

§12. Example

Let us examine a two-dimensional space of circle representations:

$$\varphi \to \varphi + a + \varepsilon \cos\varphi \equiv T_{a,\varepsilon}(\varphi). \quad (1)$$

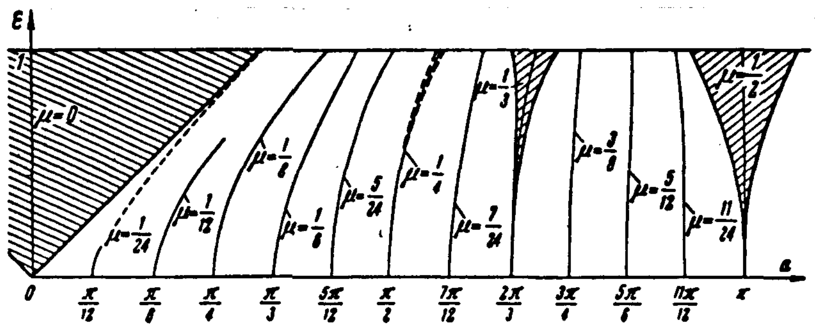

Fig. 8

With $\epsilon = 0$, we get $T_{a,0}$, a rotation to angle a. With $|\epsilon| < 1$, formula (1) defines a direct one-to-one continuous representation of a circle.

With $|\epsilon| \leq 1$, the sets of the level of the continuous function

$$\mu(a, \varepsilon) = \mu(T_{a,\varepsilon})$$

can be studied from two angles. First, it is possible to look for points (a, ϵ) of the plane where μ is rational; the boundaries of such regions are found in the semistability conditions of the cycle. For example, point (a, ϵ) is included in the set of level $\mu = 0$, if the equation

$$\varphi = \varphi + a + \varepsilon \cos\varphi$$

has a real solution, that is, the straight lines $a = \pm\epsilon$ serve as the boundary of region $\mu = 0$. The same method can be used to find the regions $\mu = \frac{m}{n}$. They approach the straight line $\epsilon = 0$ with tapering prongs (Fig. 8): the two boundaries of the $\mu = \frac{m}{n}$ have an (n - 1)-th order of tangency. The $\mu = \frac{1}{2}$ and $\mu = \frac{1}{3}$ regions, for example, have these curves as their boundaries

$$a = \pi \pm \frac{\varepsilon^2}{4} + O(\varepsilon^4), \quad (2)$$

$$a = \frac{2\pi}{3} + \frac{\sqrt{3}}{12}\varepsilon^2 \pm \frac{\sqrt{7}}{24}\varepsilon^3 + O(\varepsilon^4). \quad (3)$$

Hence we get the approximate formulas which are suitable also for not very small ϵ: when $\epsilon = 1$, formula (2) produces $\pi \pm 0.25$ instead of $\pi \pm 0.23237\ldots$.

The second approach to a definition of the sets of level $\mu(a, \epsilon)$ is through the use of the Newton method of the approximate finding of the lines of irrational level μ. After two steps by the Newton method we get an approximate equation of the level lines /79

$$a = 2\pi\mu + \frac{\varepsilon^2}{4}\operatorname{ctg}\pi\mu - \frac{\varepsilon^4}{32}\operatorname{ctg}^3\pi\mu + \frac{\varepsilon^4}{32}\operatorname{ctg}2\pi\mu(1 + \operatorname{ctg}^2\pi\mu), \quad (4)$$

which is quite effective when the cotangents are not large. Fig. 9 provides an idea of the nature of the approximation convergence and the correspondence between this result and the preceding one (this figure shows a graph of function $\mu(a) = \mu(a, 1)$; the zero approximation of the Newton method is indicated by 0, the first approximation by I, and the second by II; the horizontal sections of $\mu = 0, \frac{1}{2}, \frac{1}{3}$ are defined independently, according to formulas (2), (3)). With number a, indicated in formula (4), the change of variable

$$\psi(\varphi) = \varphi - \frac{\varepsilon}{2}\frac{\sin(\varphi - \pi\mu)}{\sin \pi\mu} + \frac{\varepsilon^2}{4}\frac{\sin(2\varphi - \pi\mu)}{\sin \pi\mu \sin 2\pi\mu}$$

changes transformation (1) into transformation

$$\psi \to \psi + 2\pi\mu + F_2(\psi, \varepsilon, \mu),$$

where $F_2 \sim \epsilon^4$.

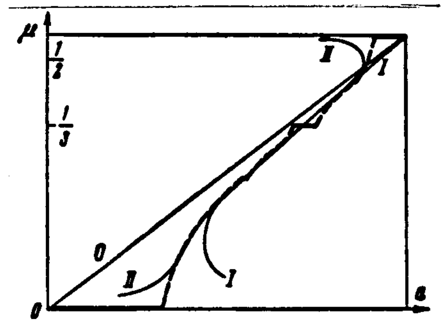

Fig. 9

Remark. The "capturing" phenomenon, corresponding to zones with rational rotation numbers, is well known in the theory of oscillation.

The transformation (1) and diagram in Fig. 8 describe the operating conditions of a generator of relaxation oscillations synchronized with sinusoidal impulses (see [36]). Another problem of a similar type, also connected with the representation of a circle, is reviewed in the book [37] (pp. 221-231).

§13. Concerning Trajectories on a Torus*

13.1. Let the following differential equation

$$\frac{dy}{dx} = F(x, y) \quad (F(x + 2\pi k,\ y + 2\pi l) = F(x, y) > 0)$$

be given on the torus x, $y \in (0, 2\pi)$, and let the usual theorem terms of existence and uniqueness of solutions be fulfilled. The following trajectory runs through each point y_0 of meridian $x = 0$

$$y(x, y_0), \quad y(0, y_0) = y_0.$$

Following Poincare, we will compare each point y_0 with point $y_1 = y(2\pi, y_0)$. We will then get a direct one-to-one representation of circle $x = 0$ which is continuous and, with a fairly smooth (or analytic) right-hand part, is smooth (or analytic); but if the function F (x, y) differs little from a constant, this representation will be close to a rotation. All the characteristics of transformation $y_1(y_0)$ reflect corresponding characteristics of the solution of equation (1), and we must only formulate the results of the preceding paragraphs in new terms. /80

*See [1]-[4], [14], [19] and [20].

If the representation y_1 (y_0), made by the substitution of $\phi(y)$ for variable y, changes to a rotation to angle $2\pi\mu$, it would be natural to extend such a substitution to the entire torus, indicating at point $(x, y(x, y_0))$

$$\varphi(x, y) = \varphi(y_0) + \mu x.$$

Obviously, if $\phi(y)$ is a smooth (and correspondingly analytic) substitution, the change of $\phi(x, y)$ on the entire torus will be the same. The trajectories will be indicated in coordinates x, ϕ as

$$\varphi = \varphi_0 + \mu x$$

which is why they speak of such a substitution as rectifying the trajectories. A. N. Kolmogorov [14] achieved an analytic rectification in the case of an analytic integral invariant. We can affirm, on the basis of theorem 2, that if function F (x, y) is analytically close to a constant, and if rotation number μ fulfills the usual arithmetical terms, the trajectories can be rectified analytically. Hence the presence of an analytic integral invariant in the dynamic system

$$\frac{dy}{dt} = F(x, y), \quad \frac{dx}{dt} = 1$$

(the invariant measure is the area in coordinates x, ϕ).

On the other hand, it is possible, as in the example of §1, to construct such an analytic function F (x, y) that the invariant measure of the system is not absolutely continuous in relation to area dx dy, even though the rotation number μ is irrational and the system ergodic.*

13.2. Let us assume the following system of differential equations

$$\frac{dx}{dt} = A(x, y), \quad \frac{dy}{dt} = B(x, y) \quad (A(x, y) > 0, \quad B(x, y) > 0) \tag{1}$$

on the torus with an analytic right-hand part. Let us take a look at equation

$$\frac{dy}{dx} = \frac{B(x, y)}{A(x, y)},$$

*Footnote to proofreading. The contrary assertion that appeared in abstract [41] during the printing of this work is erroneous. (p. 80) which has the same integral curves as the system. If they can be rectified, according to point 13.1, the system will look like this in the new coordinates

$$\frac{dx}{dt} = A'(x, \varphi), \quad \frac{d\varphi}{dt} = \mu A'(x, \varphi),$$

where $A'(x, \phi) = A(x, y(x, \phi))$. This system has an analytic integral invariant $\frac{1}{A'(x, \phi)}$, and publication [14] shows how to change it to the following system (with the usual assumptions of μ)

$$\frac{du}{dt} = 1, \quad \frac{dv}{dt} = \mu$$

by an analytic change of variables.

A contrary possibility in connection with both an equation and a system is offered by the availability of limit cycles [20]. The division of the space of the right-hand parts of system (1) into sets of the rotation number level, the segregation of rough systems and the discussion of typicalness are similar to those reviewed in §§9-11. It appears that: /81

1. The predominant topological case is that of normal cycles (which is also a crude case).* The appropriate set of right-hand parts is open and absolutely dense; but this case cannot occur in systems with an integral invariant.

2. The ergodic case (of an irrational μ) is also typical, if the evaluation of typicalness is based on measures in finite-dimensional subspaces. For systems with an analytic integral invariant this case is predominant.

13.3. In a multidimensional case lacking an integral invariant the rotation number is not defined. It is nevertheless possible to make the following assertion on the basis of the remark in 4.4.

LEMMA 9. Let $\vec{\mu} = (\mu_1, \ldots, \mu_n)$ be a vector with incommensurable components, such that with any integral vector $\vec{k}$

$$|(\vec{\mu}, \vec{k})| > \frac{C}{|\vec{k}|^n}.$$

*It is asserted in abstract [19], judging from [21], that the necessary and adequate condition for a crude case is the presence of one stable cycle. That is incorrect. (p. 81)

Then there exists such $\epsilon(R, C, n) > 0$ that for any analytic vector field $\vec{F}(\vec{x})$ on the torus (that is, such that $\vec{F}(\vec{x} + 2\pi\vec{k}) = \vec{F}(\vec{x})$), and a sufficiently small $|\vec{F}(\vec{x})| < \epsilon$ with $|\text{Im } \vec{x}| < R$, there will be found a vector a for which the differential equation system

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}) + \vec{a}$$

is changed to

$$\frac{d\vec{u}}{dt} = 2\pi\vec{\mu}$$

by an analytic change of variables.

§14. The Dirichlet Problem for the Equation of a Vibrating String

14.1. **Let D be a region on a plane which is convex in coordinate** directions, that is, its boundary Γ intersects each of the straight lines $x = c$, $y = c$ at not more than two points.

The Dirichlet problem for the equation $\frac{\partial^2 \mu}{\partial x \, \partial y} = 0$ on D is to find on it the function $u(x, y) = \phi(x) + \Psi(y)$ which is converted to Γ in a given function $f(a)$ $(a \in \Gamma)$: $u|_\Gamma = f$.

In this connection, f, ϕ, Ψ, Γ may be expected to meet various requirements in regard to smoothness, analyticity, etc.

When D is a rectangle $0 \leq x + y \leq l$, $0 \leq y - x \leq t$, it is convenient to change to the coordinates $\xi = x + y$, $\tau = y - x$. Then our equation is found to be the equation of a string, and the problem can be interpreted as finding the motion of a string by two instantaneous photographs and the end-point motion. From physical considerations (standing waves) it is clear that when l and t are commensurable, the problem is not always solvable, and if it is, it cannot be solved by a unique method. This problem is dealt with in a number of abstracts (see [22], [23], [5], [24], [17], [28]); difficulties of a similar order are encountered also in the solution of certain other problems (see [25]-[27]). /82

14.2. Uniqueness theorems (see [5]). We shall compare the boundary Γ with some of its representations (see Fig. 10). Let P be a transformation changing point $a \in \Gamma$ to point $Pa \in \Gamma$ with the same coordinate x; let Q be a transformation changing point $a \in \Gamma$ to point $Qa \in \Gamma$ with the same coordinate y. These transformations are continuous, one-to-one

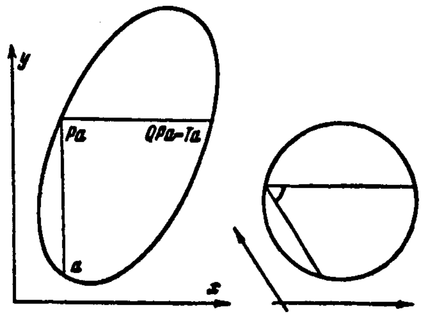

Fig. 10

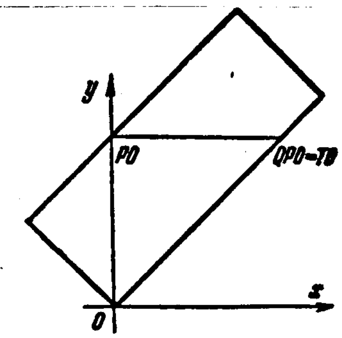

Fig. 11

and change the orientation of the Γ contour line. We will designate QP = T. Obviously,

$$P^2 = Q^2 = E, \quad PQ = T^{-1}.$$

T is a direct homomorphic representation.

THEOREM 10 (see [5]). If the contour line of Γ is such that for some point $a_0 \in \Gamma$ the set $T^n a_0$ ($n = 0, 1, 2, \ldots$) is absolutely compact on Γ, then the Dirichlet problem for Γ cannot have more than one continuous solution.

Proof. The solution $u(x, y) = \phi(x) + \Psi(y)$ defines the functions $\phi(x)$, $\Psi(y)$ correct to a constant. We will show that, by the terms of the theorem, a knowledge of $\phi(x)$ at one point $a \in \Gamma$ makes it possible to define $\phi(T^n a)$, $\Psi(T^n a)$ in all points $T^n a$ ($n = 0, 1, \ldots$) (we write $\phi(a)$ and $\Psi(a)$ to designate $\phi(x)$ and $\Psi(y)$ where x, y are the coordinates of point $a \in \Gamma$).

Knowing $\phi(a)$, it is easy to find

$$\psi(Pa) = f(Pa) - \varphi(a),$$

as the abscissas at points a and Pa are the same. It is then possible to define

$$\varphi(Ta) = f(Ta) - \psi(Pa),$$

by using the coincidence of the ordinates of points Pa and Ta. Later we shall use the same method to get ϕ, ψ in all points $T^n Pa$, $T^n a$. They form an absolutely compact set on Γ, and the continuous functions coinciding at these points therefore also coincide everywhere on Γ. The theorem has been proved.

When D is a rectangle $0 \leq x + y \leq 1$, $0 \leq y - x \leq t$, transformation T is in effect a rotation. Namely, if the following parameter is introduced in the contour line of Γ, /83

$$\vartheta = \frac{2a\pi}{\sqrt{2}(l+t)},$$

where a is the length read off from point 0 to a along the contour line (Fig. 11), then our transformation

$$T : T\vartheta = \vartheta + \frac{2\pi t}{t+l}$$

is a rotation (turn) to angle $2\pi \frac{t}{t+1}$. If D is an ellipse, it is easy to introduce a parameter in Γ so that the transformation is recorded in it as a rotation. Namely, we will take an affined mapping of an ellipse on a circle. The straight lines running in coordinate directions will become two families of parallel lines, with two straight lines of different families forming an angle $\pi\mu$ which is not necessarily a right angle. Obviously, when the ellipse is subjected to transformation T, the circle will rotate to angle $2\pi\mu$ (Fig. 10).

If Γ is a curve with a limited curvature, then T is a twice differentiable transformation; it follows, according to the Denjoy theorem, that when the rotation number μ of transformation T is irrational, the set $T^n a$ is absolutely compact on Γ. Hence

THEOREM 11 (see [5], [24]). If Γ has a limited curvature and μ is irrational, the Dirichlet problem can have only one continuous solution.

Remark. Using the theorem of the density point, it is easy to show that, by the terms of our theorem, there can be only one measurable solution. On the other hand, with μ being irrational, the method of proving theorem 10 enables us to construct any number of solutions, but mostly immeasurable ones.

14.3. A Thorough Investigation of a Rectangle.

THEOREM 12 (see [33], [17]). Let a given function f (ϑ), differentiable $p + \epsilon$ times along the boundary, be on boundary Γ of rectangle $0 \leq x + y \leq 1$, $0 \leq y - x \leq t$. Then for all $\mu = \frac{t}{t+1} \in M_k$, satisfying inequality $\left|\mu - \frac{m}{n}\right| > \frac{K}{|n|^3}$ with any m and n and some $K > 0$, the Dirichlet problem with the mentioned boundary function has a $p - 1$ times differentiable solution, and is correct in relation to f (ϑ). If f is analytic, the solution with the same μ is analytic.

In the case of some irrational μ, regardless even of the analyticity of f (ϑ), the solution may be found to be

1) only continuously differentiable,

2) differentiable k times but not k + 1 times,

3) only continuous,

4) discontinuous,

5) immeasurable.

Proof. If

$$f(\vartheta) = \sum_{n \neq 0} a_n e^{in\vartheta}, \quad \varphi(\vartheta) = \sum_{n \neq 0} b_n e^{in\vartheta}, \quad \psi(\vartheta) = \sum_{n \neq 0} c_n e^{in\vartheta},$$

then, inasmuch as $\phi(\vartheta)$ depends only on x, and $\Psi(\vartheta)$ only on y, we have: /84

$$\varphi(\vartheta) = \varphi(-2\pi\mu - \vartheta), \quad b_n = b_{-n} e^{in2\pi\mu},$$
$$\psi(\vartheta) = \psi(-\vartheta), \quad c_n = c_{-n}.$$

As f (ϑ) is real, and therefore $a_n = \bar{a}_{-n}$, we find from inequality f (ϑ) = $\phi(\vartheta) + \Psi(\vartheta)$:

$$b_n + c_n = a_n, \quad b_n e^{-in2\pi\mu} + c_n = \bar{a}_n,$$

or

$$b_n = \frac{\bar{a}_n - a_n}{e^{-2\pi i\mu n} - 1}, \quad c_n = a_n - b_n. \tag{1}$$

Now that a formal solution has been found, the proof can be completed by a verbatim repeat of the reasoning of § 2.*

Remark. Formula (1) shows that by breaking the series it is possible, in all cases of μ, to construct an "approximate solution" whose degree of approximation is the higher, the less commensurable l and t are. With a rational μ, the approximation is not higher than the side defined by μ, and when l and t are highly immeasurable, we have theorem

*Footnote to proofreading. In an article published by P. P. Mosolov [42], when this article was at the printer's, an assertion similar to theorem 12 is proved with reference to any linear differential equation with constant coefficients where the orders of all derivatives are even-numbered.

11. N. N. Vakhania refers to the correctness with respect to the region in this sense [28].

We can state that the dependence of the solution on μ is monogenic (see §7).

14.4. A general case. If boundary D is such that transformation T can be represented as a rotation in a parameter which is a smooth function of a boundary point, it is obvious that all the reasoning of 14.3. is applicable to such a contour line, and in the case of a "sufficiently irrational" μ, the Dirichlet problem has a smooth solution.

The ellipse for which a parameter was constructed in 14.2. can serve as an example. Generally, however, given an irrational μ, and regardless of the smoothness of Γ, it is impossible to guarantee that the parameter (existing according to the Denjoy theorem), in which transformation T becomes a rotation, will be smooth. F. John [5] showed that by a continuous change of variables x, y of the $x \to u(x)$ and $y \to v(y)$ type ("conserving equation $\frac{\partial^2 w}{\partial x \partial y} = 0$"), it is possible to map a region, for which T has an irrational μ, onto a rectangle or ellipse with the same μ. But this change, generally speaking, is only continuous, and it can change a smooth boundary condition on a curve to an uneven condition on an ellipse.

We should point out that if Γ is an analytic curve, then P and Q, as well as T and T^n, are analytic representations. But if Γ is also a curve analytically close to an ellipse, the transformation in the suitable parameter will be analytically close to a rotation. It therefore follows from theorem 2 that all the curves for which $\mu \in M_k$ are similar to an ellipse, in relation to the solvability of the Dirichlet problem, are at any rate fairly close to an ellipse.

The other theorems dealing with the representation of a circle can be formulated in these terms in exactly the same way. In particular, /85
if transformation T has a cycle, the Dirichlet problem with a zero boundary condition has a nonzero solution (at least a piecewise constant solution; for more details see [24]). The Dirichlet problem for the equation of a vibrating string is a problem of eigen-values for S. L. Sobolev's two-dimensional equation

$$\frac{\partial^2 \Delta u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(see [24], [27], [29], [30]). The spectrum includes the λ-values for which the representation T_λ, built on curve Γ_λ, has a cycle (here the curve Γ, subjected to a λ-dependent extension, is designated by Γ_λ).

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